Long-wavelength gravitational waves and cosmic acceleration

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Abstract

Strong long-scale gravitational waves can explain cosmic acceleration within the context of general relativity without resorting to the assumption of exotic forms of matter such as quintessence. The existence of these gravitational waves in sufficient strength to cause observed acceleration can be compatible with the cosmic microwave background under reasonable physical circumstances. An instance of the Bianchi IX cosmology is demonstrated which also explains the alignment of low-order multipoles observed in the CMB. The model requires a closed cosmology but is otherwise not strongly constrained. Recommendations are made for further observations to verify and better constrain the model.
Keywords: Dark energy, acceleration, gravitational waves, Bianchi model, CMB, Axis of Evil, CMB cold spot.

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Part I

Introduction

1 Background

The observational confirmation that the universe has been expanding from a condition of extreme density and minute size since some point in the finite past represents a major triumph of Einstein's theory of gravitation in providing an elegant explanation for cosmology, without the addition of exotic, heretofore-unobserved substances or fundamental forces. This notion has however faced a serious challenge since Riess's 1998 discovery[1] of cosmic acceleration. The purpose of this research is to evaluate the question: can the back-reaction of long-wavelength gravitational waves in a closed universe contribute to cosmic acceleration while remaining compatible with observational constraints?

1.1 Standard cosmology predicts an expanding universe

The full Einstein equations read[1]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ is the once-contracted Riemann tensor, R is the Ricci curvature scalar, $T_{\mu\nu}$ is the energy-momentum tensor, Λ is the "cosmological constant" and the constant $k \equiv 8\pi G/c^4 \approx 2.08 \times 10^{-43} \mathrm{kg}^{-1}\mathrm{m}^{-1}\mathrm{s}^2$. The Bianchi identity guarantees $T_{11} = T_{22} = T_{33}$ so, in a Gaussian ($g_{00} = 1$) and synchronous ($g_{0i} = 0$) coordinate system we have[4]:

$$\begin{aligned} R_0^0 - \frac{1}{2} R &= k T_0^0 + \Lambda && (2) \\ -R &= k T_\mu^\mu - 2\Lambda && (3) \end{aligned}$$

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[1] Throughout this document, indices written with Greek letters μ, ν etc. run over 0,1,2,3 and indices written in Roman letters i, j etc. run over 1,2,3. The sign of the metric tensor reads $+, -, -, -$.

Cosmological parameters

In discussions of cosmology it is conventional to track the expansion of an isotropic metric by introducing a "scale factor", a positive function of time only. In general the scale factor has no specific geometric meaning other than to compare distances in the metric at different points in time. Furthermore, the scale factor loses unique meaning when the universe becomes non-isotropic. We will denote this scale factor function as $a\left(t\right)$ in analogy with its definition in the Robertson-Walker metric, where it appears as[5]

$$ds^2 = dt^2 - a^2\left(t\right)\frac{\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2}{1 + \frac{1}{4}K\left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right]^2} \tag{4}$$

and the symbol K has the value 0 in a flat universe, 1 in a closed universe, and -1 in an open universe. In this case the Einstein equations read[2]

$$\frac{3}{a^2}\left(\dot{a}^2 + K\right) = k\epsilon + \Lambda \tag{5}$$

$$-6\frac{\ddot{a}}{a} = k\left(\epsilon + 3p\right) - 2\Lambda \tag{6}$$

where ϵ denotes the energy density of matter described by the energy-momentum tensor and p denotes the pressure of matter described in that tensor. Let the Hubble parameter be defined by $H \equiv \dot{a}/a$ and let H_0 be the value of H at the present time (this is how we will generally use the subscript *0*). Because the universe seems flat and dominated by ordinary matter over small scales, it is common to move terms arising from K to the right-hand side of the equation, where they act as elements of an "effective energy-momentum tensor", and to state contributors to cosmological expansion as dimensionless parameters Ω_i in comparison to the "critical density" $\epsilon_{\text{critical}}$, that is, the energy-density of ordinary matter required for the universe to be flat: $k\epsilon_{\text{critical}} = 3H_0^2$ so

$$\begin{gathered} 3H_0^2 = k\epsilon_0 + \Lambda - 3\tfrac{K}{a_0^2} \\ \Omega_K + \Omega_M + \Omega_R + \Omega_\Lambda = 1 \end{gathered} \tag{7}$$

where

$$\Omega_M + \Omega_R \equiv k\epsilon_0/3H_0^2 \tag{8}$$

$$\Omega_\Lambda \equiv \Lambda/3H_0^2 \tag{9}$$

$$\Omega_K \equiv -K/\dot{a}_0^2 \tag{10}$$

. multiple observations, most recently by WMAP, have confirmed that $\Omega_R \ll \Omega_M$[16] and so to the limit of the precision with which these quantities can be evaluated $\Omega_K + \Omega_M + \Omega_\Lambda = 1$.[3]

[2] A single dot denotes a derivative with respect to *t;* two dots denote a second derivative with respect to *t*.

[3] Chernin, in [81], elegantly derives a description of the scale factor in an open Friedmann cosmology which can be used when there is a significant amount of relativistic matter in a cold matter-dominated universe. Chernin's equation is easily generalized to the closed Friedmann universe.

1.2 Simple cosmology predicts a decelerating universe

If the scale factor a measures a distance, it is reasonable to say by analogy that $\dot{a}$ can be compared to a velocity and $\ddot{a}$ an acceleration. In isotropic cosmology we define the "deceleration parameter" Q by[4]:

$$Q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = \frac{d}{dt}\frac{1}{H} - 1 \tag{11}$$

. Dividing (6) by (7) we easily obtain

$$Q = \frac{1}{2}\frac{k\left(\epsilon + 3p\right) - 2\Lambda}{k\epsilon + \Lambda - K/a^2} = \frac{1}{2}\Omega_M - \Omega_\Lambda \tag{12}$$

. A flat universe with no cosmological constant must always decelerate. While the properties of so-called "dark matter" remain undetermined, the localisibility of dark matter's distribution and its slow motion implies it can be treated as $w = 0$ dust.

We can also immediately say that in a universe with no cosmological constant, acceleration is possible under the condition

$$\frac{2}{1+3w}\left(1 - K/a^2 k\epsilon\right) < 0 \tag{13}$$

.

1.3 Observations say the universe is accelerating

Acceleration in and of itself is not a newcomer to cosmology. The de Sitter cosmology[82], discovered in 1917, is driven solely by a cosmological constant and consequently has a constant deceleration parameter of $Q = -1$. Bondi, Gold & Hoyle's "steady state" universe[83] similarly accelerates with $Q = -1$, this value being associated with a universe whose expansion is driven solely by a field whose energy density is not dependent on the size of the universe. With the proposal of "big bang" nucleosynthesis[84] and the subsequent discovery of the cosmic microwave background[85], consensus came to settle on the simplest matter-filled model, the Friedmann universe.[86]

Throughout the 1990s, astronomical observations began to indicate that the matter energy density of the universe was far below the critical density, leading some (for example [87]) to propose the resurrection of the cosmological constant in order to preserve the observed near-flatness of space.

In 1998, Riess *et al.* published an analysis[1] of the light from a small number of type Ia supernovae with $0.16 \leq z \leq 0.62$ and concluded from this set that the recent universe is accelerating with $Q_0 = -1.0 \pm 0.4$. Further observations and analysis (see Part II) have also provided evidence that the universe has $Q_0 < 0$.

[4] Q has been defined with a minus sign for historical reasons. $Q > 0$ denotes a decelerating universe; $Q < 0$ denotes an accelerating universe. We have avoided the more common notation q in favor of Q to avoid confusion when interpreting the source material.

While Riess *et al.* did not exclude the possibility of a universe with $K \neq 0$, the assumption of a flat universe remains predominant throughout the field of cosmology as observations, both from supernova data and WMAP, have shown that the universe is, on observable scales, very close to flat – although it is impossible to distinguish between a universe that is genuinely flat, with $\Omega_K = 0$ and one with Ω_K very close to but not equal to zero.

2 Dark energy

Since the discovery of acceleration, numerous explanations for the phenomenon have been proposed, all depending on an isotropic field creating additional, invisible energy. Turner and Huterer[6] introduce the term "dark energy", analogous to dark matter in the sense that dark energy does not interact electromagnetically with ordinary matter and has the property of an energy density, to term this additional energy, which appears to make up over 70% of the total energy content of the universe[16].

The assumption of a flat homogeneous cosmology demands that cosmic acceleration comes from a cosmological constant or a scalar field. Most scalar theories for explaining cosmic acceleration fall into two classes: an exotic form of matter with negative energy density, and surrender of the cosmological principle. Other scalar theories sacrifice different assumptions, such as homogeneity, or invoke more exotic explanations unsupported by laboratory physics.

2.1 Cosmological constant

The simplest most familiar variation on the Robertson-Walker cosmological model which allows an accelerating universe is the "ΛCDM" model – a universe dominated by "cold" (non-relativistic, $p = 0$) matter with both baryonic and dark components, and with the existence of a non-zero cosmological constant. In such a universe the Einstein equations read[5]

$$3H^2 = k\epsilon + \Lambda \tag{14}$$

$$-6\frac{\ddot{a}}{a} = k\epsilon - 2\Lambda \tag{15}$$

so when $k\epsilon/\Lambda$ is small such that $(k\epsilon/\Lambda)^2$ is negligible, that is, the universe is dominated by a cosmological constant,

$$Q = \frac{1}{2}\frac{k\epsilon - 2\Lambda}{k\epsilon + \Lambda} \approx -1 + \frac{3k\epsilon}{2\Lambda} \tag{16}$$

which at first glance appears to neatly explain Riesse *et al.*'s result. However, as will be shown (see SECTION 6), the case for a cosmological constant is not definite. Furthermore, the theoretical background explaining the strength of the cosmological constant is not well developed, relying on an understanding of quantum gravity which does not yet exist[72]. While the cosmological constant can always be said to have a "right to exist" in the Einstein equations, current physics does not explain why it should have any particular strength and as such the cosmological constant should be treated as the simplest form of a scalar field of exotic matter.

2.2 Quintessence

More general than the cosmological constant but similar in structure is the proposal of "quintessence"[6], a novel form of matter with a time-dependent equation of state that can take on negative values. Many forms of these have been proposed; one form of these, for example, is the "Chaplygin gas"[88], which has equation of state $p = -A/\epsilon$ for $A > 0$. Quintessence theories are particularly motivated by the idea that acceleration is a cosmologically recent phenomenon, noting limited data (see PART II) that the equation of state of dark energy may be evolving with time.

Any formulation of quintessence must be regarded as highly speculative. At the most fundamental level, all theories of quintessence propose the existence of a kind of matter which:

- has never been observed experimentally;
- does not interact with ordinary matter via the electromagnetic force;
- has a negative equation of state, that is, a positive energy density produces a negative pressure;
- plays a prominent role at current energy levels, as opposed to exotic effects (e.g. unification of forces) thought to have taken place only in the very early universe.

In the absence of any compelling experimental evidence whatsoever for any kind of quintessence, quintessence and quintessence-like models should be rejected as definitive explanations for dark energy.

2.3 Local inhomogeneity

A more mundane explanation which has been offered for acceleration is the "Hubble bubble"[1, 61], regions of lower density in the intergalactic medium. If the vicinity of the Milky Way had lower matter energy density, expansion in its vicinity would increase[93], causing the illusion of cosmic acceleration.

Not only would the density deficit in such a "bubble" have to be quite large in order to cause acceleration, but the theory, which has the advantage of requiring no new physics, supposes either the existence of a rare or unique void that the Milky Way happens to be in – a violation of the cosmological principle in the sense that it makes observers in the Milky Way privileged – or a preponderance of voids whose presence makes the universe inhomogeneous not just in small patches but on average.[91, 94]

2.4 Exotic models

Modified relativity

Some proposals to explain dark energy propose modifications to the Einstein equations. The best-known of these is the Cardassian Expansion model[95], which proposes time-dependent variation of the equation of state of matter. The Cardassian model is of particular interest in that it proposes an equation for the density perturbation

$$\kappa''(x) + 2\frac{s}{x}\kappa' - \frac{3}{2}s^2\kappa = 0 \tag{17}$$

for unknown constant s, which equation begins to resemble that for weak gravitational waves in a closed universe (cf. EQUATION (153)). Like Chaplygin gas and the "DGP" model[96], the Cardassian model justifies itself based on theories about higher-dimensional manifolds which remain untested.

Topological defects

The existence of cosmic strings would change the overall equation of state of the matter in the universe by a constant[67, 97], creating acceleration through simple deviation from the Friedmann model. While theories of cosmic inflation predict the formation of cosmic strings and other topological defects, such defects remain completely undetected.

3 Tensorial theories for acceleration in a flat universe

If we wish to preserve the theory of relativity and cosmic homogeneity, while at the same time relying only on effects with good experimental basis, scalar fields appear to be excluded as an explanation for acceleration. Ergo within the context of general relativity the next place to search for an answer is in tensor theories, which include the possibility of gravitational waves.

Lifshitz's theory of cosmological perturbations[79] appears to exclude tensorial answers to the problem of acceleration: gravitational waves have the same equation of state as radiation, and local clumps gravitational waves in the theory (where "local" means bounded within an region smaller than the radius of curvature of the universe) both decay rapidly and collapse spatially. Rodrigues[113] takes a first step in discussing anisotropic dark energy, but limits his analysis to a flat universe and thus creates the problem of an anisotropic "big rip".

A high-frequency gravitational wave background has been proposed[92] as the source of cosmic acceleration. While the authors' analysis appears initially promising, similar to many scalar dark energy candidates the theory relies on the existence of an inflation-induced gravitational wave background that remains only hypothetical. Furthermore, the authors obtain their result by selection of an averaging scheme without mathematical rigor – surely choosing a mathematical model based on the desired results cannot be considered physics. At any rate, the strength of the background inflationary theory predicts is not sufficiently great to explain the observed large acceleration.

Part II

Evidence for acceleration

4 Introduction

The theory of tests to evaluate the deceleration parameter using supernovae as standard candles began with Wagoner[45] in 1977. Starting from assumptions of an isotropic Friedmannian cosmology which is not necessarily flat, Wagoner notes the approximate relation

$$d_E = H_0^{-1}\left[z - \frac{1}{2}\left(1+Q_0\right)z^2 + \mathcal{O}\left(z^3\right)\right] \tag{18}$$

which, when H_0 and z are known, relates the deceleration parameter to the distance d_E as determined by the dimming of the supernova (where Wagoner was originally discussing Type II supernova events)[5]. This relation is valid when z is small such that z^3 is negligible, limiting its usefulness above $z \sim 1$, and requires the assumption of only small changes in the Hubble constant H_0 (that is, in a Friedmann cosmology $\dot{a}_F/a_F$ evaluated near the observer) on the interval from $z \approx 0$ to $z \approx 1$.

Type Ia supernovae are thought to be a "standard candle" for the measurement of distance and redshift; that is, supernovae of that type are thought to possess spectral and luminosity curves which are nearly identical. Therefore, observation of extragalactic type Ia supernovae is believed to produce reliable information on both the distance of the event (noting that brightness diminishes as the inverse square of distance) and the redshift of the distance associated with the event (through the change in the peak of the supernovae's spectra), with redshift z related to the scale factor a_F by

$$z + 1 = \frac{a_F\left(t_{\text{observation}}\right)}{a_F\left(t_{\text{emission}}\right)} \tag{19}$$

. Analysis of a statistically unbiased dataset of $z\left(t\right)$ therefore gives empirical information on $Q\left(t\right)$.

Colgate[44] proposed that Type I supernovae should be used to measure the deceleration parameter in preference to Type II supernovae. Type I supernovae, specifically the "Type Ia" whose mechanism is thought to be the accretion of matter onto the surface of a white dwarf star, are understood to have a well-defined typical absolute magnitude and spectrum, and assuming this is true the distance to and redshift of a given Type Ia supernova event (SNe) can easily be determined by fitting its light curve to standard templates. Therefore, with a sufficient sample of extragalactic supernovae of $z \lesssim 1$, the parameters H and Q can be measured directly. When an isotropic cosmology with constant deceleration parameter $Q = Q_0$ is assumed, knowledge of H_0 and Q_0 are sufficient to typify the parameters of the universe[4].

With the advent of modern optical astronomy such as adaptive optics[46] and space-based optical telescopy[47], such surveys have become possible, but have produced results contradicting the standard, cold matter-filled Friedmann model of cosmology.

[5] Equation (18) is of course a generalization of the famous distance-redshift approximation $H_0 d_E \approx z$.[4]

5 Surveys of acceleration

Cosmological studies measuring Q have been ongoing since 1997 and consist of analysis of redshifts[1, 3, 9, 11, 12, 13, 15, 43, 48, 49, 51, 50, 52, 53, 55, 56, 54, 57, 58] of type Ia supernovae.

The High-z Supernova Search Team's initial study of the deceleration parameter[1] was the first large study to call attention to the problem of acceleration. Working from a sample of sixteen supernovae (four of which were well-observed "high confidence" sources), the most distant with $z = 0.97$, Riess concluded that the universe has $Q_0 < 0$ to high confidence, although the measurement of Q_0 itself possessed a high degree of uncertainty. Riess also noted the high sensitivity of the result to individual data points. Oddly, the authors dismiss the closed cosmology despite the data indicating it as preferred[1, Fig. 7]; however the size of their experimental error precludes real evaluation of spatial curvature.

The Supernova Cosmology Project (SCP) made an earlier attempt to evaluate Q_0 with the use of supernovae[55]. This small survey ($n = 7$) on relatively nearby supernovae found a result inconsistent with those that followed it, giving results consistent with a universe with no dark energy and with too high a degree of error to meaningfully evaluate the geometry of the universe.

In contrast, the Supernova Cosmology Project's 1998 evaluation[48, 49] of 42 Type Ia SNe added further evidence that the universe was accelerating, and also makes note of the surprising coincidence of the energy density Ω_Λ's near-equivalence with the total energy density in the current epoch. The SCP also failed to consider the closed cosmology despite supernova data favoring it[49, Fig. 7].

The ESSENCE[11] survey was expressly designed to examine cosmic acceleration and detected 102 type-Ia supernovae from $0.10 \leq z \leq 0.78$, of which 60 were used for cosmological analysis. The initial analysis of ESSENCE assumed flatness of the universe. ESSENCE's observational fields were deliberately chosen to overlap the areas of previous surveys and to lie within ten degrees of the celestial equator; all were also between 23:25 and 02:33 Right Ascension. Combining data from ESSENCE, SNLS and other sources[52] led to a conclusion consistent with other analyses. Exploration of more exotic models[53] found that no model of those tested was a good fit for ESSENCE's data.

The Supernova Legacy Survey (SNLS)[12] recorded 472 type-Ia supernovae. While analysis of the SNLS dataset[13] provides results consistent with a universe driven by cosmological constant, the uncertainty on analysis of a time-dependent component to the equation of state of dark energy is very large; their analysis also does not consider a closed universe as a possible model[50]. Furthermore, the SNLS team also note the presence of two outliers and only 125 of 472 events were used to evaluate cosmology. SNLS observed SNe in four fields, one of which (field 3) is far above the plane of the celestial equator at 52 degrees declination; this and [54]'s northern field are the only fields with multiple observations in a small area more than 20 degrees from the celestial equator surveyed to date. SNLS also notes[50, section 5.4] that the values of Ω_M evaluated in the four fields are compatible only at a 37% confidence level – a surprising result given that each SNLS field contains at least 60 SNe in quite small (one square degree) areas.

The Hubble Space Telescope or HST survey of supernovae, published in 2004 and reviewed by the Supernova Cosmology Project[14] observed twenty type-Ia supernovae with redshifts $0.63 < z < 1.42$. While the number of SNe observed is small, the HST survey has the advantage of covering a wider area of sky than other SNe surveys. Analysis of the HST dataset suggests a rapidly-evolving dark energy field, although with very high error on measurements greater than $z = 1$ due to the small ($n = 10$) sample size it is impossible to take these results as anything more than suggestive. HST slightly favored a closed model of the universe, when considering

interpretations of data that allowed $\Omega_K \neq 0$.

The Supernova Cosmology Project's 2008 analysis of supernova data[51] made a analysis of combined SNLS, ESSENCE and HST data, and attempted to analyze the data in the context of a theory of a time-dependent equation of state for dark energy but concluded "present SN data sets do not have the sensitivity to answer the questions of whether dark energy persists to $z > 1$, or whether it had negative pressure then." The analysis rejected 10% of all SNe from the combined data sets as outliers, many based on their failure to fit with a nearby H_0; Kowalski *et al.*'s rejection of outliers also shifts their analysis from one favoring a closed universe to one favoring a flat one[51, Fig. 11].

Further work by Riess *et. al.*[54, 56] produced the so-called "gold" dataset of SNe, a group of supernova events with particularly clear light curves with 33 at $z > 1$. These supernovae were observed in two small (one square degree) fields. [54] claims a great reduction in the uncertainty of the Hubble parameter at $z > 1$ but the Hubble parameter measured in the extended "gold" set gives a value for the Hubble parameter not reconcilable with that in the [56] dataset. Riess *et. al.* conclude that w is negative (with large experimental error) in the region $1 < z < 2$, then attempt to extrapolate the behavior of dark energy back to $z = 1089$.

Sollerman *et al*'s analysis of the Sloan Digital Sky Survey-II supernova data[3, 9] is the most recent analysis indicating cosmic acceleration. SDSS-II observed 103 type-Ia supernovae in a long, narrow strip along the celestial equator, including many from lower redshifts than had been previously examined in detail; Sollerman *et. al.* also made use of data from the HST, ESSENCE and SNLS surveys, bringing the total number of SNe examined to 288. The primary conclusion to be drawn from SDSS is the sensitivity of cosmological measurements to the specific analysis technique used[60]; analysis of the data with two different curve-fitting algorithms produce two different, albeit somewhat compatible, results.

Further obscuring the neatness of measuring Q, Jha *et. al.* noted[59] that the uneven local distribution of galaxies, specifically the existence of voids, can lead to a mis-estimation of H_0 on the order of 6.5% for a given galaxy.

Finally, of note is the WiggleZ dark energy survey[15, 43]. WiggleZ is the most extensive redshift survey thus far conducted, with some 280,000 galaxies with $0.2 < z < 1.0$ used as sources. WiggleZ also covers a wider area of sky than previous surveys, examining some 1000 square degrees in multiple windows around the sky. Two of WiggleZ's windows overlap with SDSS-II's survey area, so while WiggleZ is ongoing preliminary results [57, 58] can be used to improve the evaluation of Q by improving precision on measurements of z of SNe host galaxies. The authors of [58] note that "the redshift-space clustering pattern is not isotropic in the true cosmological model", attributing the variation to "the coherent, bulk flows of galaxies toward clusters and superclusters". Analysis by the WiggleZ team of pre-existing SNe datasets, using the new, more precise data on galaxy redshifts they obtained, reconfirms the fact of acceleration, and generates results consistent with other surveys, but the data lack sufficient precision to determine the history of Q.

Table 3 details the sky locations of SNe and galaxies used in the determination of acceleration; Figure 5 presents these locations graphically. Table 1 in Appendix summarizes the results of these surveys.

Survey	No of SNe	z
Supernova Cosmology Project 1997[55]	7	$0.35 < z < 0.46$
High-z Supernova Search Team[1]	16	$0.16 < z < 0.97$
Supernova Cosmology Project 1998[49]	42	$0.18 < z < 0.86$
HST[14]	20	$0.63 < z < 1.42$
ESSENCE[11]	102	$0.10 \leq z \leq 0.78$
Supernova Legacy Survey[12, 50]	125	$0.015 < z < 1$
ESSENCE + SNLS[52]	162	$0.015 < z < 1$
Supernova Cosmology Project combined[51]	307	$0.015 < z < 1$
Riess "gold" sample[56, 54]	16	$1.25 < z < 2$
WiggleZ[58]	557	$0.1 < z < 0.9$

Survey	$\Omega_\Lambda^\dagger$	Ω_M	Ω_K
Supernova Cosmology Project 1997	$0.06^{+0.28}_{-0.34}$	$0.94^{+0.34}_{-0.28}$	(dne)
High-z Supernova Search Team	$0.72^{+0.72}_{-0.48}$	$0.24^{+0.56}_{-0.24}$	(dne)
Supernova Cosmology Project 1998	$0.72^{+0.08}_{-0.09}$	$0.28^{+0.09}_{-0.08}$	(dne)
HST	$0.715^{+0.036}_{-0.057}$	$0.286^{+0.022}_{-0.023}$	$-0.001^{+0.037}_{-0.013}$
ESSENCE	$0.726^{+0.020}_{-0.032}$	$0.274^{+0.032}_{-0.020}$	(dne)
Supernova Legacy Survey	0.751 ± 0.080	0.271 ± 0.020	(dne)
ESSENCE + SNLS	$0.733^{+0.018}_{-0.028}$	$0.267^{+0.028}_{-0.018}$	(dne)
Supernova Cosmology Project combined	$0.785^{+0.046}_{-0.045}$	$0.285^{+0.030}_{-0.030}$	$-0.010^{+0.016}_{-0.015}$
Riess "gold" sample	$0.71^{+0.03}_{-0.05}$	$0.29^{+0.05}_{-0.03}$	(dne)
WiggleZ	0.71 ± 0.03	0.29 ± 0.03	(dne)
Sloane Digital Sky Survey-II	$0.693 \pm .042^1$ $0.744 \pm .041^2$ $0.74 \pm .02^3$	$0.307 \pm .042^1$ $0.256 \pm .041^2$ $0.25 \pm .02^3$	$.04 \pm .04$

Survey	$Q_0^{\text{flat}\ddagger}$	w_{X0}	w_{Xa}
Supernova Cosmology Project 1997	$0.41^{+0.14}_{-0.17}$	(dne)	(dne)
High-z Supernova Search Team	-1.0 ± 0.4	(dne)	(dne)
Supernova Cosmology Project 1998	$-0.58^{+0.04}_{-0.05}$	(dne)	(dne)
HST	$-0.572^{+0.045}_{-0.025}$	$-0.997^{+0.266}_{-0.293}$	$0.13^{+1.16}_{-1.57}$
ESSENCE	$-0.589^{+0.022}_{-0.004}$	$-1.047^{+0.125}_{-0.124}$	(dne)
Supernova Legacy Survey	-0.620 ± 0.060	-1.023 ± 0.087	(dne)
ESSENCE + SNLS	$-0.600^{+0.009}_{-0.014}$	$-1.069^{+0.091}_{-0.093}$	(dne)
Supernova Cosmology Project combined	$-0.642^{+0.030}_{-0.031}$	$-1.001^{+0.149}_{-0.155}$	(dne)
Riess "gold" sample	$-0.56^{+0.05}_{-0.02}$	$-1.02^{+0.13}_{-0.19}$	*
WiggleZ	-0.56 ± 0.02	(dne)	(dne)
Sloane Digital Sky Survey-II	$-0.539 \pm .021^1$ $-0.616 \pm .021^2$ $-0.61 \pm .01^3$	$-0.76 \pm .18^1$ $-0.96 \pm .18^2$	(dne)

Table 1: Summary of results from surveys indicating acceleration

*"dne" = "Does not evaluate". *: [56] attempts to analyze w_a with several different constraints but provides no numerical figure for its estimate of w_a's value. †: Where not explicitly stated in the source, Ω_Λ is evaluated from $\Omega_M + \Omega_K + \Omega_\Lambda = 1$. ‡: $Q_0^{flat} = \frac{1}{2}\Omega_M - \Omega_\Lambda$. (1): MLCS2K2 evaluation. (2): SALT-II evaluation. (3): ΛCDM model evaluation.*

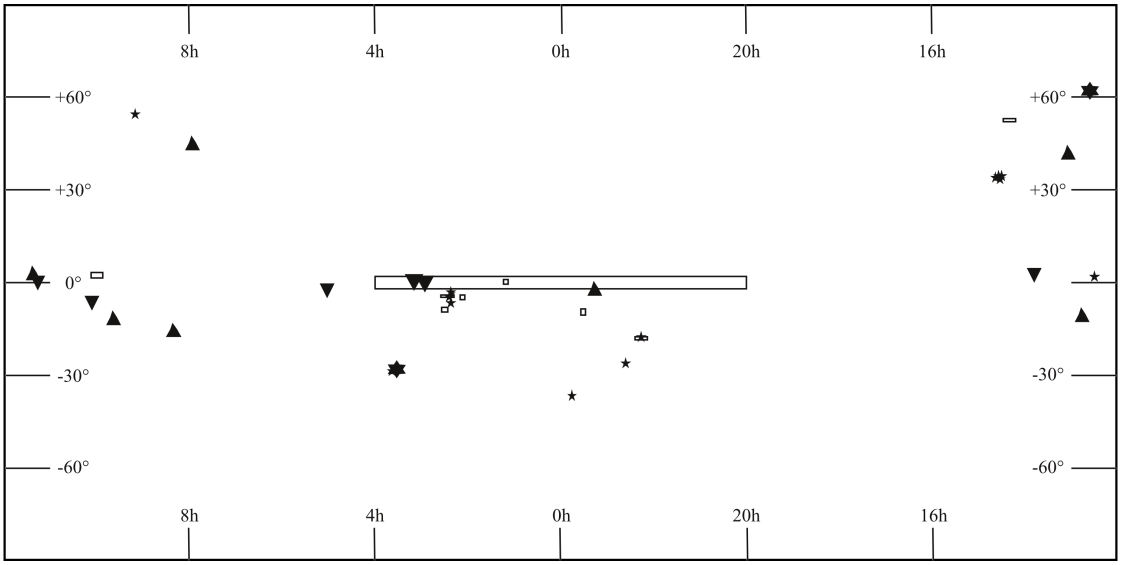

Figure 1: Sky positions of supernovae used as evidence for acceleration
Surveys of cosmic acceleration cover a limited portion of the sky, and data are divided into two contiguous, antipodal regions. Most data has been collected in a small area of the sky near the equator. *Triangles: Riess 1998 supernovae. Five-pointed stars: HST SNe. Six-pointed stars: Riess "gold" dataset. The long, thin strip centered on 0,0 is the SDSS-II survey area. Other boxes are the SNLS and ESSENCE survey areas.*

6 Analysis

Analysis of supernova data is, in one sense, quite consistent: all surveys apart from [55] agree that for $z < 1$ we have a deceleration parameter $Q_0 = -0.6$. Deeper analysis suffers from a lack of data at high redshifts and large numbers of free parameters in cosmological models, especially when more exotic models are considered. Meanwhile, while most surveys indicate that the acceleration in recent times acts as though driven by a cosmological constant, with an equation of state compatible with $w_X = -1$, the results from [60] show that this can be the result of the prior assumptions made about the model of dark energy.

No definitive statement can be made about the evolution of Q_0 over time from the information thus far available, particularly not statements connecting the state of cosmic acceleration now with the state of acceleration at the epoch of last scattering.

Nor can any definitive statement be made about cosmological models, other than to say that the most conservative, ΛCDM model fits the data at best inconsistently. Few studies of supernova data on acceleration examine the question of curvature in depth.

The majority of SNe data is collected from a single patch of sky: the field bounded by RA 22:00, RA 04:00, Dec $+1°15'$ and Dec $-10°00'$ (the "highly-observed field"). This area comprises 1350 square degrees, or only 2.1% of the sky. Surveys taken in small fields outside the highly-observed field, such as the Riess "gold" dataset, have high internal consistency, while surveys covering larger areas of sky have much lower consistency; the "gold" dataset contains the same number of SNe as the [1] but has a standard error less than a tenth the size. It is also telling that the four SNLS fields produced results that correlated poorly (37% confidence) with one another[50], where two of the SNLS survey regions are well outside the highly-observed field. Compounding cosmographic

bias, many of the remaining SNe observations are located in a region of sky antipodal from the highly-observed field; any vector or tensor contribution to cosmic dynamics will be dominated by dipole and quadrupole terms, and as such be seen with equal or opposite magnitude in the antipodal direction (that is, if we observe a change in Q of ΔQ along the x^idirection, we should expect a change of $-\Delta Q$ in the event of a vector contribution, or ΔQ in the event of a tensor contribution, along the $-x^i$ direction).

There is, furthermore, no SNe data whatsoever from above Dec $+62°$or below Dec $-37°$. The authors of [58] note a variation in the apparent Hubble parameter for galaxies in this equatorial band (no WiggleZ region lies further north than Dec $+8°$ or Dec $-19°$); variation to the Hubble flow could potentially be even greater outside this region. There is also no evaluation of whether the Hubble flow remains isotropic beyond $z = 0.3$ [77].

Indeed, Zehavi *et. al.* comment[89] on the lack of sky coverage in their analysis of local Hubble flows, noting that "sparse sampling and the incomplete sky coverage (especially at low Galactic latitudes) may introduce a bias in the peculiar monopole due to its covariance with higher multipoles". While the fact of greater redshift in the range where acceleration can be measured should overcome the peculiar velocities of galaxies, the data problem remains.

7 Conclusions

Many reasonable constraints prevent a full-sky survey of supernovae. In the optical band, much of the sky is obscured by the "zone of avoidance" created by the plane of our own galaxy[63]. The so-called "Great Attractor", certain to be a region of particularly high peculiar velocities and therefore great shifts in the apparent Hubble parameter, lies in this zone[64]. Furthermore, with only a single space-based optical observatory (the *Hubble Space Telescope*) operating, detailed observation of the sky is restricted to those latitudes accessible by ground-based observatories, none of which are located in Arctic latitudes. However, the directional deficit of SNe surveys, aggregated together, cannot be ignored.

In the light of Tegmark *et. al.*'s discovery[28] of a preferred axis to the CMB quadrupole, and Land & Maguiejo's subsequent observation[32] of a preferred axis in higher multipole moments aligned with the the quadrupole (the so-called "Axis of Evil"), the default assumption should be that anisotropic acceleration is not ruled out. Indeed, the prominent CMB "Cold Spot"[34] falls within the highly-observed field, although no surveys or SNe are located exactly in its direction.

As such, Wagoner's assertion of the cosmological principle as "statistically valid"[45] has been misapplied by analysts of acceleration data. A tensorial theory of cosmic acceleration would preserve homogeneity, in the sense that every observer sees "the same version of cosmic history"[21], at the expense of isotropy in the form of spherical symmetry.

More fundamentally, all studies of cosmic acceleration to date operate on the assumption that acceleration is isotropic, that is, that the acceleration field is equal in every direction, and therefore must be explained either by a cosmological constant or a scalar field. As Mörtsell and Clarkson note, "[a]t best this gives a small error to all our considerations; at worst, many of our conclusions might be wrong"[61]. In particular, the data as presented cannot distinguish between a scalar-field theory of acceleration, a vector-field theory of acceleration, a cosmological constant theory of acceleration, and a time-dependent tensor-field theory of acceleration.

Meanwhile, the simplest theory of acceleration, a cosmological constant, is challenged on two fronts: not only is Ω_Λ's value far out of line with that predicted by theory[72], but while its

equation of state is close to $w_X = -1$ measurements have tended to favor a value slightly smaller than -1.

It is interesting to note that when Ω_K is evaluated, supernova data favor a closed universe (although always in a manner compatible with a flat universe); this conclusion is consistent with the curvature parameter evaluated by WMAP[62].

7.1 Recommendations

In light of these weaknesses of the current information on cosmic acceleration, the following program is recommended:

The data already in hand should be re-evaluated to look for signs of angular dependence in the Hubble parameter. In particular, the SNLS fields 2 and 3 and the "gold" sample fields should be examined in and of themselves in order to build up a map of H as a function of both z and direction in the sky. The rejection of certain SNe in [51] should be re-evaluated in light of possible inadvertent obscuring of evidence for angular dependence in H.

Analyses of SNe data should always consider the possibility of a closed or open universe as well as a flat one.

Additional SNe surveys for redshifts $.15 < z < 2$ should be carried out in unexamined areas of sky not obscured by the plane of the galaxy, such as for example the celestial north and south poles. The optimal region for these surveys is in rings located 90° from the center of the highly-observed field, which will maximize the difference in the event of a tensor-field (that is, gravitational-wave) acceleration.

In light of this need and the lack of ground-based observatories, as well as the Zone of Avoidance, priority should be given to the Wide Field Infrared Survey Telescope ("WFIRST") project[65], which incorporates the Super Nova/Acceleration Probe[66, 67] and Joint Dark Energy Mission[68, 69]. This telescope is currently scheduled to be launched in 2016.

As WiggleZ continues, its data on galactic redshifts should be examined for angular dependence as well. The completion of WiggleZ will provide invaluable information on baryon acoustic oscillations which will make possible the charting of the history of H and Q at much higher redshifts than is possible through the examination of supernova data.

Zhao *et al.* have also noted the possibility of using the Einstein telescope as an instrument for examining dark energy through the use of gravitational wave emissions from colliding binary objects as a "standard siren" analogous to the standard candle of type Ia SNe[111]. The largest binary systems, in the period prior to merger, could be parameterized by radio telescopy[116].

Cooray and Caldwell[102], implicitly identifying the same problem of lack of angular coverage as we note herein, propose a program of near-redshift surveys covering a large but practical area of sky which could also provide the relevant information with existing facilities.

Overall, the need is underscored for new theories of acceleration, particularly ones that attempt to explain acceleration through the action of tensor perturbations in a closed universe. Wagoner's formula (EQUATION 18) and its generalizations must be generalized further, to take into account the possibility of anisotropic fields as the cause of anisotropic cosmic acceleration.

Part III

The Bianchi IX cosmology

In pursuit of a theory within the context of unmodified general relativity which can explain cosmic acceleration while remaining compatible with the cosmic microwave background, we wish to relax as few constraints on our cosmological model as necessary. Therefore while having sacrificed the requirement of isotropy in the sense of spherical symmetry in the dark energy field, we wish to retain a stronger[2, ss. 116] condition of the Copernican principle on our space, that of homogeneity[4, Chap. 13 sec. 1]. It is also desirable to have a model whose limiting case is a Friedmann cosmology, in order to explain the almost-isotropic (that is, almost-Friedmannian) character of the CMB. Furthermore, a model which is spatially closed, in order to match models favored by CMB and SNe data, is desirable; such a model would, if complying with all other conditions, have a flat universe as a limiting case in the limit of an infinitely large radius of curvature.

Bianchi showed[70] that there exists exactly one homogeneous[6] space with a closed Friedmann universe as a limiting case: the Bianchi type IX cosmology, for short "Bianchi IX".

8 The Bianchi classification scheme

Bianchi observed that all three-dimensional homogeneous spaces could be classified into nine types, based on categorization of the symmetries, that is the Killing field, in each space. Behr noted[71] that this categorization scheme could be simplified to filling a parameter space of four parameters: one running over the real numbers and three reducible to the sign function $\operatorname{sgn}(x)$.

Consider some space with metric $ds^2 = dt^2 - g_{ij}dx^i dx^j$ (that is, a space in Gaussian coordinates) where $g_{ij} = g_{ij}\left(t, x^i\right)$. If the sub-space with metric tensor g_{ij} is homogeneous, then there exists a set of vectors that solve $\xi_{i;j} + \xi_{j;i} = 0$; these are the Killing vectors of the space[4]. In a homogeneous space, these vectors will (where [,] is a commutator) obey the commutation relationship

$$[\xi_i, \xi_j] \equiv \xi_i\xi_j - \xi_j\xi_i = C^k_{ij}\xi_k \tag{20}$$

where in an homogeneous space, the object C^k_{ij} is a constant pseudo-tensor, the "structure constants" of an homogeneous space, with the antisymmetry property $C^k_{[ij]} = C^k_{ij}$ [2, ss. 116].

We always have the freedom to perform separation of variables the functions g_{ij}; let us do so by defining the matrix γ_{ab} such that

$$g_{ij}\left(t, x^k\right) = -\gamma_{ab}\left(t\right) e^{(a)}_i\left(x^k\right) e^{(b)}_j\left(x^k\right) \tag{21}$$

[6] A homogeneous space is a space such that for any two points in the space, there exists a geodesic, not necessarily of finite length, connecting those two points.

.[7] The 3×3 matrix $e_i^{(a)}\left(x^k\right)$ is a triad[2, 112, ss. 98][8] of vectors ("frame vectors") which solve

$$\xi_k\left(e_i^{(a)} e_j^{(b)} dx^i dx^j\right) = 0 \tag{22}$$

(in the language of linear algebra, the quantity $e_i^a dx^i$ is a one-form on a homogeneous space).

Furthermore, define the matrix $e_{(a)}^i$ such that $e_{(a)}^i = e_j^{(a)} = \delta_j^i$; from this it follows that $e_{(a)}^i e_i^{(b)} = \delta_{(a)}^{(b)}$. From these relationships we can transform between any tensor and its decomposition into triads by saying that for some tensor $A_{j_1 j_2 j_3 \ldots j_n}^{i_1 i_2 i_3 \ldots i_m}$,

$$A_{j_1 j_2 j_3 \ldots j_n}^{i_1 i_2 i_3 \ldots i_m} = A_{(b)_1 (b)_2 (b)_3 \ldots (b)_n}^{(a)_1 (a)_2 (a)_3 \ldots (a)_m}\left(e_{(a)_1}^{i_1} e_{(a)_2}^{i_2} e_{(a)_3}^{i_3} \ldots e_{(a)_m}^{i_m}\right)\left(e_{j_1}^{(b)_1} e_{j_2}^{(b)_2} e_{j_3}^{(b)_3} \ldots e_{j_n}^{(b)_n}\right) \tag{23}$$

; therefore in an homogeneous space we can perform separation of variables on the partial differential equations of general relativity and solve the time-dependent parts as ordinary differential equations.

The frame vectors obey the properties

$$e_{i,j}^{(a)} - e_{j,i}^{(a)} = C_{bc}^a e_i^{(b)} e_j^{(c)} \tag{24}$$

[7]. The structure constants C_{bc}^a typify a homogeneous space and are given by the following rule[71][9]:

$$C_{bc}^a = \varepsilon_{bcd} n^{ad} + \delta_c^d a_b - \delta_b^d a_c \tag{25}$$

where the object n^{ab} is a diagonal matrix diag $\left(n^{(1)}, n^{(2)}, n^{(3)}\right)$ and a_a is the vector $(a, 0, 0)$, the values of this matrix and vector typifyied by the underlying cosmology (Table 2).

The cosmologies of Bianchi types I, V, VII_0, VII_a and IX are of particular interest as they have isotropic spaces as limiting cases; specifically, a universe with metric

$$ds^2 = dt^2 - a^2 \eta_{ab} e_i^{(a)} e_j^{(b)} \tag{26}$$

is a flat $K = 0$ universe for Bianchi type I, an open $K = -1$ universe for Bianchi types V, VII_0, VII_a, and a closed $K = 1$ universe for Bianchi type IX[2, 10, 21]. Bianchi IX is the only homogeneous closed cosmological model in the context of general relativity[104].

[7] Indices from the beginning of the Latin alphabet (a, b, c,...) denote triad indices; indices from the middle of the alphabet (i, j, k,...) denote regular indices. Where the two are mixed or the application is otherwise ambiguous, triad indices are enclosed in parentheses; in this work, this notation never means the tensor symmetrization operation.

[8] The widespread *Fourth Revised English Edition* of [2] contains numerous serious typographical errors in the section introducing the tetrad formalism. The Russian-language *Seventh Corrected Edition*[112] contains the correct formulas.

[9] The symbol ε_{abc} represents the Levi-Civita symbol defined such that $\varepsilon_{123} = 1$

Bianchi type	a	$n^{(1)}$	$n^{(2)}$	$n^{(3)}$
I	0	0	0	0
II	0	1	0	0
III	1	0	1	-1
IV	1	0	0	1
V	1	0	0	0
VI_0	0	0	1	-1
VI_a	a	0	1	-1
VII_0	0	1	1	0
VII_a	a	1	1	0
VIII	0	1	1	-1
IX	0	1	1	1

Table 2: The Bianchi classification scheme

Constants for the different homogeneous spaces of the Bianchi classification scheme[2, 21, 71, 104]. The quantity a runs over the real numbers. This parametrization is not unique (we could, for example, have chosen $(-1,-1,-1)$ for $\left(n^{(1)}, n^{(2)}, n^{(3)}\right)$ in the type IX space).

9 The Kasner universe

In order to illustrate the possible effects of an anisotropic but homogeneous cosmology on cosmic dynamics, we will consider a Bianchi type I cosmology that generalizes the Friedmann cosmology: the Kasner universe[76]; [2, ss. 117].

Let our metric read

$$ds^2 = dt^2 - t^{2p_1}\left(dx^1\right)^2 - t^{2p_2}\left(dx^2\right)^2 - t^{2p_3}\left(dx^3\right)^2 \tag{27}$$

where p_1, p_2, p_3 are constants. In a co-moving coordinate system we quickly arrive at the following set of Einstein equations:

$$\left[(p_1+p_2+p_3) - \left(p_1^2+p_2^2+p_3^2\right)\right] t^{-2} = \frac{1}{2}k\,(\epsilon+3p) \tag{28}$$

$$(p_1+p_2+p_3-1)\,p_1 t^{-2} = \frac{1}{2}k\,(p-\epsilon) \tag{29}$$

$$(p_1+p_2+p_3-1)\,p_2 t^{-2} = \frac{1}{2}k\,(p-\epsilon) \tag{30}$$

$$(p_1+p_2+p_3-1)\,p_3 t^{-2} = \frac{1}{2}k\,(p-\epsilon) \tag{31}$$

. These equations necessitate either an isotropic but unphysical $(p=\epsilon)$ universe or a vacuum $(\epsilon = p = 0)$ universe, in which we have either the trivial solution $p_1 = p_2 = p_3 = 0$ (Minkowski space) or the more interesting solution

$$p_1+p_2+p_3 = p_1^2+p_2^2+p_3^2 = 1 \tag{32}$$

. This solution admits a parametrization of p_1, p_2, p_3 such that (if we choose $p_1 \le p_2 \le p_3$)

$$\begin{aligned} p_1 &= -u/\left(1+u+u^2\right) \\ p_2 &= (1+u)/\left(1+u+u^2\right) \\ p_3 &= u(1+u)/\left(1+u+u^2\right) \end{aligned} \tag{33}$$

where $u > 0$; these relations have the nice symmetry property that $p_i(u) = p_i(1/u)$.

9.1 Scale factor

The scale factor a does not necessarily have an intrinsic meaning, but instead compares distances as a function of time. In an isotropic cosmology such as the Friedmann model a can be given a real geometric meaning; in an open or closed Friedmann universe, the scale factor appears simply in the Ricci curvature of space $R^i_j = \left(2K/a^2\right)\delta^i_j$ and as such can be regarded as the radius of curvature of the universe. In particular, in a closed isotropic universe a can be considered to have the direct physical meaning of the radius of curvature of the spherical space, so in a closed universe one could meaningfully say "the radius of the universe is a".

When space is no longer isotropic, the definition of scale factor breaks down. It is, of course, possible to define any positive function as "the" scale factor. Grishchuk *et al.*[10, section 4], for example, use a metric

$$\begin{aligned} \gamma_{11} &= \frac{1}{4}a^2 e^{2\alpha} \\ \gamma_{22} &= \frac{1}{4}a^2 e^{2\beta} \\ \gamma_{33} &= \frac{1}{4}a^2 e^{2\gamma} \end{aligned} \tag{34}$$

and propose the definition

$$a^2 \equiv \frac{1}{12}\gamma_{ab}\eta^{ab} \tag{35}$$

in the context of a vacuum cosmology, motivated by the coincidence of this definition of the scale factor with one the authors introduce in separating the Bianchi IX metric into background and gravitational-wave parts. The authors also discuss a definition of scale factor such that

$$a^2 \equiv \left(\det \gamma_{ab}\right)^{1/3} \tag{36}$$

. This definition has the advantage that it relates the scale factor to a definite physical quantity, a volume element, but it contains a deeper flaw: with such a definition in place the Einstein equations admit no solution other than the background solution at quadratic and higher orders. If we define the quantity

$$\delta \equiv \alpha + \beta + \gamma \tag{37}$$

then

$$a^2 \equiv (\det \gamma_{ab})^{1/3} \implies e^{\delta} = 1 \implies \delta = 0 \tag{38}$$

. In either case, though, discussion of definitions of a attempt to solve a problem that does not exist. The question of what definition of scale factor to select is analogous to the question of which of the orthocenter, incenter or circumcenter of a triangle is the "true" center. Consequently, attempting to extract a single scale factor – and thus a single Hubble parameter or a single deceleration parameter – from anisotropic Einstein equations is a fool's errand.

We can, if we wish, split the metric (34) into isotropic and anisotropic parts by noting that the quantity $a_F e^{\delta}$ is isotropic and that any two of the quantities $\alpha - \beta$, $\alpha - \gamma$ and $\beta - \gamma$ combined with $a_F e^{\delta}$ contain all the information needed to describe the metric[22]; pursuing this route would be a distraction from our main task.

Instead, let the notion of scale factor a, Hubble parameter H and deceleration parameter Q be generalized. In a homogeneous cosmology with a diagonal metric, define the following matrices: the generalized scale factor,

$$a_{ab} \equiv \begin{pmatrix} (\gamma_{11})^{1/2} & 0 & 0 \\ 0 & (\gamma_{22})^{1/2} & 0 \\ 0 & 0 & (\gamma_{33})^{1/2} \end{pmatrix} \tag{39}$$

(recalling that non-integer powers of a matrix are not defined, so we could not simply say $a_{ab} \equiv (\gamma_{ab})^{1/2}$). In a Bianchi I cosmology only, from this definition we can then define the redshift matrix (in homogeneous cosmologies other than Bianchi I the geodesic equations are non-linear; see PART IV):

$$z_a^b \equiv a_{ac}(\eta_R)\, a^{bc}(\eta_E) - \delta_a^b = \begin{pmatrix} \frac{a_{11}(t_R)}{a_{11}(t_E)} - 1 & 0 & 0 \\ 0 & \frac{a_{22}(t_R)}{a_{22}(t_E)} - 1 & 0 \\ 0 & 0 & \frac{a_{33}(t_R)}{a_{33}(t_E)} - 1 \end{pmatrix} \tag{40}$$

where the subscript R denotes the function evaluated at the time of observation of light, and E denotes the function evaluated at the time of emission; and finally the generalized Hubble parameter and deceleration parameter:

$$H_{ab} \equiv \frac{1}{2}\frac{d}{dt}\ln \gamma_{ab} = \begin{pmatrix} \dot{a}_{11}/a_{11} & 0 & 0 \\ 0 & \dot{a}_{22}/a_{22} & 0 \\ 0 & 0 & \dot{a}_{33}/a_{33} \end{pmatrix} \tag{41}$$

$$Q_a^b \equiv \frac{d}{dt}H^{ac}\eta_{bc} - \delta_b^a = -\begin{pmatrix} \ddot{a}_{11}a_{11}/\left(\dot{a}_{11}\right)^2 & 0 & 0 \\ 0 & \ddot{a}_{22}a_{22}/\left(\dot{a}_{22}\right)^2 & 0 \\ 0 & 0 & \ddot{a}_{33}a_{33}/\left(\dot{a}_{33}\right)^2 \end{pmatrix} \tag{42}$$

. This approach is essentially a generalization of that developed by Barrow in [22]; the object (41) is closely related to the shear tensor[21, 26]. The practical purpose of these definitions is to

provide a mathematical description of observed quantities; let $\mathbf{e}^i$ be a unit vector pointing in the direction of observation. Then the redshift observed in the $\mathbf{e}^i$ direction is given by

$$z\left(e^i, t\right) = z^a_b e^i_{(b)} e^{(a)}_j \mathbf{e}^j \mathbf{e}_i \tag{43}$$

and similarly for other functions of the scale factor. Each of these functions can be averaged over the whole sky, these averages denoted by a bar:

$$\bar{a} \equiv \frac{\int a_{ab} e^{(b)}_i e^{(a)}_j \mathbf{e}^i \mathbf{e}^j dS}{\int \eta_{ij} \mathbf{e}^i \mathbf{e}^j dS} = \frac{1}{3} a_{ab} \eta^{ab} = \frac{1}{3}\left(a_{11} + a_{22} + a_{33}\right) \tag{44}$$

etc.

9.2 Dynamics in the Kasner universe

An observer in a Kasner universe will see the consequences of that universe's evolution. Examination of the observational consequences of the Kasner universe provides an illustrative example of potential consequences of anisotropy in other cosmologies.

Expansion

Misner, Thorne & Wheeler argue[76] that the Kasner universe is expanding, as the volume element is always increasing:

$$\frac{dV}{dt} = \frac{d}{dt}\sqrt{\|g_{ij}\|} dx^1 dx^2 dx^3 = \frac{d}{dt}\left(t^{p_1+p_2+p_3}\right) dx^1 dx^2 dx^3 = dx^1 dx^2 dx^3 \tag{45}$$

. However, as noted above there is no unique way to define the scale factor. In terms of the averaged quantity defined in (44) we have

$$\bar{a} = \frac{1}{3}\left(t^{p_1} + t^{p_2} + t^{p_3}\right) \tag{46}$$

which, when we expand around $t = 1$, is approximately

$$\bar{a}\left(t \approx 1\right) \approx \frac{1}{3}\left(2 + t\right) + \mathcal{O}\left(t^3\right) \tag{47}$$

. But in the limit of t small, we have

$$\bar{a} \approx \frac{1}{3} t^{p_1} \tag{48}$$

, which is clearly a decreasing function; so the Kasner universe is not unambiguously expanding.

Redshift

Redshift in a Kasner universe is given by

$$z^i_j = \begin{pmatrix} (t_R/t_E)^{p_1} - 1 & 0 & 0 \\ 0 & (t_R/t_E)^{p_2} - 1 & 0 \\ 0 & 0 & (t_R/t_E)^{p_3} - 1 \end{pmatrix} \tag{49}$$

$$\bar{z} = \frac{1}{3}\left[\left(\frac{t_R}{t_E} - 1\right)^{p_1} + \left(\frac{t_R}{t_E} - 1\right)^{p_2} + \left(\frac{t_R}{t_E} - 1\right)^{p_3}\right] \tag{50}$$

. In the circumstance when $t_R \gg t_E$,

$$\bar{z} \approx \frac{1}{3}\left(\frac{t_R}{t_E}\right)^{p_3} \tag{51}$$

. Of particular interest is the quantity $\Delta T/T_R$, the variation in CMB temperature from the average (accepting for the moment that the vacuum Kasner universe approximates a matter-filled one at a sufficiently young age), which is given approximately by

$$\begin{aligned} \frac{\Delta T}{T_R} &\approx \left[3\left(\frac{t_R}{t_E}\right)^{-p_3}\begin{pmatrix} (t_R/t_E)^{p_1} & 0 & 0 \\ 0 & (t_R/t_E)^{p_2} & 0 \\ 0 & 0 & (t_R/t_E)^{p_3} \end{pmatrix} - \eta_{ab}\right]\mathbf{e}^i\mathbf{e}^j = \\ &= \begin{pmatrix} 3\,(t_R/t_E)^{p_1-p_3} - 1 & 0 & 0 \\ 0 & 3\,(t_R/t_E)^{p_2-p_3} - 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\mathbf{e}^i\mathbf{e}^j \approx \\ &\approx \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\mathbf{e}^i\mathbf{e}^j \end{aligned} \tag{52}$$

(except in the case when $p_2 = p_3 = 2/3$, in which event the (2,2) entry in (52) will read 2). The CMB in a mature Kasner universe has a pronounced anisotropy, with the observed temperature matching the average temperature only in a circle around the axis of anisotropy. Notably, the primary axis of the anisotropy is at a right angle to the axis along which the Kasner universe is contracting – not on a parallel axis!

Hubble flow & deceleration parameter

The Kasner universe has Hubble flow

$$H_{ab} = \frac{1}{t}\begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix} \tag{53}$$

$$\bar{H} = \frac{1}{3} t^{-1} \tag{54}$$

and deceleration parameter

$$Q_a^b = \begin{pmatrix} (1-p_1)/p_1 & 0 & 0 \\ 0 & (1-p_2)/p_2 & 0 \\ 0 & 0 & (1-p_3)/p_3 \end{pmatrix} \tag{55}$$

$$\bar{Q} = -1 \tag{56}$$

. In the limit that the parameter $u \to \infty$ an observer in a Kasner universe would see a universe with a positive Hubble flow (redshift) over most of the sky, but see blueshift in a third direction. An observer looking only at averages, though, would not be able to distinguish between an isotropic universe and a Kasner universe merely by examining the Hubble flow; only with a complete picture of the sky is such a test possible. The Hubble flow in the case of minimal anisotropy has the form

$$H_{ab}(u=1) = \frac{1}{t} \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix} \tag{57}$$

– appearing like a Friedmannian matter-dominated universe in two directions – and in the case of maximal isotropy

$$\lim_{u\to 0} H_{ab} = \frac{1}{t} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{58}$$

Similarly, an observer looking only at the averaged deceleration parameter sees a universe accelerating as though driven by a cosmological constant; only with good enough information will the observer notice a strong angular dependence in the acceleration field, which in the case of minimal anisotropy has the form

$$Q_a^b(u=1) = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \tag{59}$$

– decelerating like a Friedmann cosmology in two directions – and in the case of maximal anisotropy has the form

$$\lim_{u\to\infty} Q_a^b = \begin{pmatrix} -\infty & 0 & 0 \\ 0 & \infty & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{60}$$

. Moreover, even though acceleration along two axes is negative in the least-anisotropic Kasner universe, the impact of the positive-accelerating direction is such that the isepitach[10] of zero acceleration, the boundary an observer sees on the sky between regions where objects accelerate and objects decelerate, is a circle 83° from the axis of acceleration; only less than 8% of the sky appears close to "normal" to an observer expecting to record a Friedmann universe!

While the vacuum Kasner universe is ruled out as a possible cosmology both for reasons of the CMB, which appears isotropic to a high degree[16], and due to the Hubble flow, which appears isotropic to the limit of the peculiar motions of galaxies below $z = 0.3$ [77], the surprising difficulties in distinguishing between its dynamics and that of a Friedmann universe serve as a reminder that sampling of cosmological parameters must be done in an unbiased fashion and that isotropy must be tested rather than assumed. The Kasner universe also has an application as a limiting case of the BKL universe[74], to which it appears identical for observers looking over a period of time that is small compared to the radius of curvature of the universe. Finally, the anisotropic Kasner universe serves as a limiting case to some types of cosmology described by the Bianchi IX model.

10 Gravitational wave nature of Bianchi IX

The Bianchi IX has been considered by cosmologists repeatedly since the establishment of general relativity to provide possible explanations for cosmological phenomena.

Belinsky, Khalatnikov and Lifshitz discussed[74] a Bianchi IX cosmology (the "BKL cosmology") which undergoes several "bounces" as it evolves – rather than expanding from or converging to a point, it contracts along one axis while expanding along two others until the smallest metric component reaches a minimum value, at which point the axes swap roles. Misner[73] discussed a related form of Bianchi IX universe as the "mixmaster universe", pursuing an resolution to the horizon problem through the non-linearity of the Bianchi IX cosmology; through the mechanism of bounces, all parts of the universe are brought into causal connection. Bouncing vacuum cosmologies are, like the vacuum Kasner universe, intrinsically highly anisotropic; while in the long run they tend to act isotropically due to the back-reaction of matter[75, 76] they will still exhibit strong CMB anisotropy[19]. Supernova data ([1, 48] *etc.*) and CMB data on the value of Ω_M ([31] *etc.*) coupled with the existence of high-redshift objects[72] rule out bouncing cosmologies to a high degree of confidence.

The BKL cosmology undergoes anisotropic acceleration (see SECTION 9.2). Meanwhile, numerical modeling has suggested[98] that a matter-filled Bianchi IX universe will also undergo periods of acceleration. Therefore, we have good reason to suppose that a property of Bianchi IX may be to generate anisotropic acceleration, and that consequences of the Bianchi IX cosmology may reveal a dark energy candidate with none of the failings of scalar or exotic models.

Wheeler showed[78] that an almost-isotropic Bianchi IX universe admitted a weak tensorial perturbation that took the form of a wave (that is, solving an equation of the form $\ddot{f} + nf(t) = g(t)$). Grishchuk *et. al.* were able to generalize this result[10]:

The Bianchi IX space has frame vectors

[10] A neologism denoting a path of constant acceleration, similar to "isobar" or "isochor", from Greek *"epitachounse"*, acceleration.

$$
\begin{aligned}
e_i^1 &= \left(\cos x^3, \quad \sin x^1 \sin x^3, 0\right) \\
e_i^2 &= \left(-\sin x^3, \quad \sin x^1 \cos x^3, 0\right) \\
e_i^3 &= \left(0, \quad \cos x^1, 1\right)
\end{aligned}
\tag{61}
$$

. Consider the metric of a Bianchi IX cosmology:

$$
ds^2 = dt^2 - \gamma_{ab} e_i^a e_j^b dx^i dx^j \tag{62}
$$

. We can split this metric up into an isotropic (Friedmannian) part and a non-Friedmannian part:

$$
\begin{aligned}
ds^2 =& dt^2 - a_F^2 \eta_{ab} e_i^a e_j^b dx^i dx^j - \left(\gamma_{ab} - a_F^2 \eta_{ab}\right) e_i^a e_j^b dx^i dx^j = \\
=& ds_0^2 - \left(\gamma_{ab} - a_F^2 \eta_{ab}\right) e_i^a e_j^b dx^i dx^j
\end{aligned}
\tag{63}
$$

. Grishchuk, Doroshkevich & Iudin showed that the object describing the space part of the anisotropic part of the metric at some moment in time,[11]

$$
G_{ij}^{ab} \equiv 2\left(e_i^a e_j^b + e_i^b e_j^a\right) - \frac{4}{3} \eta^{ab} \eta_{cd} e_i^c e_j^d \tag{64}
$$

, obeys the property

$$
\left(G_{ij}^{ab}\right)_{;k}^{;k} = -\left(n^2 - 3K\right) G_{ij}^{ab} \tag{65}
$$

for $n = 3$ and $K = 1$; that is, G_{ij}^{ab} is a tensor eigenfunction of the Laplace operator in a Bianchi IX space for waves with wavenumber $n = 3$. A similar property for open spaces is true of the Bianchi type VII models.[19][12]

Lifshitz, in his development of the theory of cosmological perturbations[75, 79, 2, ss. 115], claims that tensorial perturbations, including gravitational waves, can only have diminishing effect over time. Lifshitz is, however, considering only the class of *local* tensorial perturbations.

In contrast, the gravitational waves in Bianchi IX will have wavelengths comparable to the radius of curvature of the universe. Kristian and Sachs note[25] that the wavelength of cosmic shear (and thus, all else being equal, of cosmological gravitational waves) must be at least 2×10^{10} years – longer than the Hubble radius[16] – and could potentially be far longer (see SECTION 17).

We will consider first the regime of weak gravitational waves, in an almost-isotropic universe, and then "quasi-isotropic" waves: that is, the regime in which components of the metric evolve at equal powers of t.

[11] a_F has been scaled here to equal 1

[12] We could also choose to interpret Bianchi I as the degenerate case of a flat universe containing gravitational waves of infinite wavelength with $n = 0$. The Kasner universe, however, is *not* such a universe: all the anisotropy is governed by a single parameter, u, so the system has an insufficient number of degrees of freedom. The Kasner universe is more like the Taub universe[108].

10.1 Einstein equations in the tetrad formalism

For a metric $g_{\alpha\beta}$, let the space-space part of the metric be decomposed as in (21). Similarly, the tensors

$$R_{ij} = R_{ab} e_i^a e_j^b \tag{66}$$
$$T_{ij} = T_{ab} e_i^a e_j^b \tag{67}$$

with all space dependence in the frame vectors. Therefore the Einstein equations can be rewritten:

$$R_{00} = kT_{00} - \frac{1}{2} kT g_{00} \tag{68}$$
$$R_{0i} = kT_{0i} - \frac{1}{2} kT g_{0i} \tag{69}$$
$$R_{ab} = k\left(T_{ab} - \frac{1}{2} T \gamma_{ab}\right) \tag{70}$$

. If we have energy-momentum tensor

$$T_{\mu\nu} = (p+\epsilon)\, u_\mu u_\nu - p g_{\mu\nu} \tag{71}$$
$$T = \epsilon - 3p \tag{72}$$

then

$$T_{00} = (p+\epsilon)\, u_0 u_0 - p g_{00} \tag{73}$$
$$T_{0i} = (p+\epsilon)\, u_0 u_i - p g_{0i} \tag{74}$$
$$T_{ab} = (p+\epsilon)\, u_a u_b - p \gamma_{ab} \tag{75}$$

. If we then choose a synchronous Gaussian reference system, as we always have freedom to do,

$$g_{00} = 1 \tag{76}$$
$$g_{0i} = 0 \tag{77}$$

so the Einstein equations read

$$R_{00} = k\,(p+\epsilon)\, u_0 u_i k\,(p+\epsilon)\, u_0 u_0 - kp g_{00} - \frac{1}{2} k\,(\epsilon - 3p) \tag{78}$$
$$R_{0i} = k\,(p+\epsilon)\, u_0 u_i \tag{79}$$
$$R_{ab} = k\,(p+\epsilon)\, u_a u_b - kp\gamma_{ab} - \frac{1}{2} k\,(\epsilon - 3p)\,\gamma_{ab} \tag{80}$$

. If we then demand that our coordinate system be co-moving with matter,

$$u^0 = 1 \tag{81}$$
$$u^i = 0 \tag{82}$$

then

$$R_{00} = \frac{1}{2}k\left(\epsilon + 3p\right) \tag{83}$$
$$R_{0i} = 0 \tag{84}$$
$$R_{ab} = \frac{1}{2}k\left(p - \epsilon\right)\gamma_{ab} \tag{85}$$

. Let

$$d_{ab} \equiv \frac{1}{2}\frac{\partial}{\partial t} g_{ij} e^i_a e^j_b = \frac{1}{2}\frac{d}{dt}\gamma_{ab} \tag{86}$$

and

$$d \equiv d_{ab}\gamma^{ab} \tag{87}$$

. The Christoffel symbols associated with our metric then become [2, ss. 97]

$$\Gamma^0_{00} = \Gamma^0_{0i} = \Gamma^i_{00} = 0 \tag{88}$$
$$\Gamma^0_{ij} = d_{ij} \tag{89}$$
$$\Gamma^i_{0j} = d^i_j \tag{90}$$
$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} \tag{91}$$

where $\tilde{\Gamma}^i_{jk}$ are the Christoffel symbols associated with the three-dimensional metric tensor $-g_{ij}$. The Ricci tensor can then be written as[2, ss. 97]:

$$R_{00} = -\dot{d} - d^b_a d^a_b \tag{92}$$
$$R_{0i} = 0 \tag{93}$$
$$R_{ab} = \dot{d}_{ab} + d d_{ab} - 2 d_{ac} d^c_b - P_{ab} \tag{94}$$

or explicitly[10]

$$\dot{d} + d^b_a d^a_b = -\frac{1}{2}k\left(\epsilon + 3p\right) \tag{95}$$
$$\dot{d}_{ab} + d d_{ab} - 2 d_{ac} d^c_b - P_{ab} = \frac{1}{2}k\left(\epsilon - p\right)\gamma_{ab} \tag{96}$$
$$d^b_a C^{\ a}_{bc} = 0 \tag{97}$$

where P_{ij} is the three-dimensional Ricci tensor constructed from $\tilde{\Gamma}^i_{jk}$.

10.2 The curvature tensor for Bianchi IX

Grishchuk explicitly gives the curvature tensors for all Bianchi types, and a general method for easily deriving them, in [7]. These tensors can be stated in removable and non-removable parts, the removable parts corresponding to time-dependent rotations of the space. Let the symbol

$$\gamma_{abc} \equiv \gamma_{ad} C^a_{bc} \tag{98}$$

. Then where

$$\Gamma^c_{\ ab} \equiv \frac{1}{2}\gamma^{cd}\left(\gamma_{abd} + \gamma_{dab} - \gamma_{bda}\right) \tag{99}$$

(these are analogous to the Christoffel symbols of the full space, but with different symmetry properties) the non-removable part of the curvature tensor is given by

$$L_{ab} \equiv -2\Gamma^c_{\ a[b,c]} + 2\Gamma^c_{\ d[b}\Gamma^d_{\ |a|c]} + 2\Gamma^c_{\ ad}\Gamma^d_{\ [bc]} \tag{100}$$

where square brackets around the indices indicate the antisymmetric part of the tensor; and the removable part is given by

$$b_{ab} \equiv \frac{1}{2} v_c C^c_{\ ba} + \frac{1}{2}\left(f_a v_b - f_b v_a\right) \tag{101}$$

and finally the curvature tensor

$$P_{ab} = L_{ab} - b_{bc} d^c_a - b_{ac} d^c_b - b_{ba} d \tag{102}$$

. In the co-moving case that $v_a = 0$ we can simply state $P_{ab} = H_{ab}$. For the particular case of Bianchi IX (the frame vectors (61)) and the curvature tensor when $v_a = 0$ reads, for diagonal components:

$$P^b_a = \left[\frac{\left(\gamma_{fg}\eta^{fg}\right)^2}{2\left\|\gamma_{cd}\right\|} - \gamma^{fg}\eta_{fg}\right]\delta^b_a - \gamma^{bc}\eta_{ac} - \frac{\gamma_{af}\gamma_{gh}\eta^{fg}\eta^{bh}}{\left\|\gamma_{cd}\right\|} \tag{103}$$

and for non-diagonal components

$$P^b_a = -2\gamma^{cb}\eta_{ac} - \frac{1}{\left\|\gamma_{df}\right\|}\gamma_{ac}\gamma_{df}\eta^{bc}\eta^{df} \tag{104}$$

where $\|\gamma_{ab}\|$ is defined as the determinant of γ_{ab}. The Einstein equations show that when $v_a = 0$ the non-diagonal components of γ_{ab} must be zero, so as a consequence of our Gaussian choice of coordinate system therefore we can without loss of generality write the metric for Bianchi IX

$$\begin{aligned} \gamma_{11} &= a_F^2 e^{2\alpha} \\ \gamma_{22} &= a_F^2 e^{2\beta} \\ \gamma_{33} &= a_F^2 e^{2\gamma} \end{aligned} \tag{105}$$

with all other space-space components zero, so explicitly the the curvature tensor P_{ab} for Bianchi IX reads

$$P_{11} = \frac{1}{2} e^{-2\delta} \left(-e^{4\alpha} + \left(e^{2\beta} - e^{2\gamma} \right)^2 \right) e^{2\alpha} \tag{106}$$

$$P_{22} = \frac{1}{2} e^{-2\delta} \left(-e^{4\beta} + \left(e^{2\gamma} - e^{2\alpha} \right)^2 \right) e^{2\beta} \tag{107}$$

$$P_{33} = \frac{1}{2} e^{-2\delta} \left(-e^{4\gamma} + \left(e^{2\alpha} - e^{2\beta} \right)^2 \right) e^{2\gamma} \tag{108}$$

$$P_{ab} = 0, \qquad a \neq b \tag{109}$$

and the contracted curvature scalar

$$P_{ab}\gamma^{ab} = 2a_F^{-2} e^{-2\delta} \left[e^{4\alpha} + e^{4\beta} + e^{4\gamma} - 2 \left(e^{2\alpha+2\beta} + e^{2\beta+2\gamma} + e^{2\alpha+2\gamma} \right) \right] \tag{110}$$

.

11 Einstein equations for Bianchi IX

11.1 Exact equations

Let the symbol $\delta \equiv \alpha + \beta + \gamma$ for convenience as in (37). For our chosen metric, we have the auxiliary quantities

$$d_{11} = \left(a\dot{a} + a^2\dot{\alpha} \right) e^{2\alpha} \tag{111}$$

$$\dot{d}_{11} = \left(\dot{a}^2 + a\ddot{a} + 4a\dot{a}\dot{\alpha} + a^2\ddot{\alpha} + 2a^2\dot{\alpha}^2 \right) e^{2\alpha} \tag{112}$$

$$d_1^1 = H + \dot{\alpha} \tag{113}$$

$$d = 3H + \dot{\delta} \tag{114}$$

and cyclic permutations in α, β, γ thereof for 22- and 33-quantities. The full Einstein equations for Bianchi IX read

$$\left\{ \begin{array}{c} \frac{3}{a_F^2}\left(\dot{a}_F^2+1\right)+\dot{\alpha}\dot{\beta}+\dot{\alpha}\dot{\gamma}+\dot{\beta}\dot{\gamma}+2\frac{\dot{a}_F}{a_F}\dot{\delta}+ \\ +a_F^{-2}e^{-2\delta}\left[\begin{array}{c} 2\left(e^{2\alpha+2\beta}+e^{2\alpha+2\gamma}+e^{2\beta+2\gamma}\right)- \\ -e^{4\alpha}-e^{4\beta}-e^{4\gamma}-3e^{2\delta} \end{array} \right] \end{array} \right\} = k\epsilon \tag{115}$$

$$\left\{ \begin{array}{c} \frac{\ddot{a}_F}{a_F}+2\frac{\dot{a}_F^2}{a_F^2}+\frac{2}{a_F^2}+\ddot{\alpha}+\frac{\dot{a}_F}{a_F}\left(3\dot{\alpha}+\dot{\delta}\right)+\dot{\alpha}\dot{\delta}+ \\ +2a_F^{-2}e^{-2\delta}\left[e^{4\alpha}-\left(e^{2\beta}-e^{2\gamma}\right)^2-e^{2\delta}\right] \end{array} \right\} = \frac{1}{2}k\left(\epsilon-p^{(1)}\right) \tag{116}$$

$$\left\{ \begin{array}{c} \frac{\ddot{a}_F}{a_F}+2\frac{\dot{a}_F^2}{a_F^2}+\frac{2}{a_F^2}+\ddot{\beta}+\frac{\dot{a}_F}{a_F}\left(3\dot{\beta}+\dot{\delta}\right)+\dot{\beta}\dot{\delta}+ \\ +2a_F^{-2}e^{-2\delta}\left[e^{4\beta}-\left(e^{2\gamma}-e^{2\alpha}\right)^2-e^{2\delta}\right] \end{array} \right\} = \frac{1}{2}k\left(\epsilon-p^{(2)}\right) \tag{117}$$

$$\left\{ \begin{array}{c} \frac{\ddot{a}_F}{a_F}+2\frac{\dot{a}_F^2}{a_F^2}+\frac{2}{a_F^2}+\ddot{\gamma}+\frac{\dot{a}_F}{a_F}\left(3\dot{\gamma}+\dot{\delta}\right)+\dot{\gamma}\dot{\delta}+ \\ +2a_F^{-2}e^{-2\delta}\left[e^{4\gamma}-\left(e^{2\alpha}-e^{2\beta}\right)^2-e^{2\delta}\right] \end{array} \right\} = \frac{1}{2}k\left(\epsilon-p^{(3)}\right) \tag{118}$$

. We can also define quantities as components of a gravitational effective energy-momentum tensor:

$$k\epsilon_g \equiv -\left\{ \begin{array}{c} \dot{\alpha}\dot{\beta}+\dot{\alpha}\dot{\gamma}+\dot{\beta}\dot{\gamma}+2\frac{\dot{a}_F}{a_F}\dot{\delta}+ \\ +a_F^{-2}e^{-2\delta}\left[\begin{array}{c} 2\left(e^{2\alpha+2\beta}+e^{2\alpha+2\gamma}+e^{2\beta+2\gamma}\right)- \\ -e^{4\alpha}-e^{4\beta}-e^{4\gamma}-3e^{2\delta} \end{array} \right] \end{array} \right\} \tag{119}$$

$$\frac{1}{2}k\left(\epsilon_g-p_g^{(1)}\right) \equiv -\left\{ \begin{array}{c} \ddot{\alpha}+\frac{\dot{a}_F}{a_F}\left(3\dot{\alpha}+\dot{\delta}\right)+\dot{\alpha}\dot{\delta}+ \\ +2a_F^{-2}e^{-2\delta}\left[e^{4\alpha}-\left(e^{2\beta}-e^{2\gamma}\right)^2-e^{2\delta}\right] \end{array} \right\} \tag{120}$$

$$\frac{1}{2}k\left(\epsilon_g-p_g^{(2)}\right) \equiv -\left\{ \begin{array}{c} \ddot{\beta}+\frac{\dot{a}_F}{a_F}\left(3\dot{\beta}+\dot{\delta}\right)+\dot{\beta}\dot{\delta}+ \\ +2a_F^{-2}e^{-2\delta}\left[e^{4\beta}-\left(e^{2\gamma}-e^{2\alpha}\right)^2-e^{2\delta}\right] \end{array} \right\} \tag{121}$$

$$\frac{1}{2}k\left(\epsilon_g-p_g^{(3)}\right) \equiv -\left\{ \begin{array}{c} \ddot{\gamma}+\frac{\dot{a}_F}{a_F}\left(3\dot{\gamma}+\dot{\delta}\right)+\dot{\gamma}\dot{\delta}+ \\ +2a_F^{-2}e^{-2\delta}\left[e^{4\gamma}-\left(e^{2\alpha}-e^{2\beta}\right)^2-e^{2\delta}\right] \end{array} \right\} \tag{122}$$

$$kp_g^{(1)} \equiv \left[\begin{array}{c} 2\ddot{\alpha}+6\frac{\dot{a}_F}{a_F}\dot{\alpha}+2\dot{\alpha}^2+\dot{\alpha}\dot{\beta}+\dot{\alpha}\dot{\gamma}-\dot{\beta}\dot{\gamma}+ \\ +a_F^{-2}\left(\begin{array}{c} 5e^{2(\alpha-\beta-\gamma)}-3e^{2(\beta-\alpha-\gamma)}-3e^{2(\gamma-\alpha-\beta)}+ \\ +6e^{-2\alpha}-2e^{-2\gamma}-2e^{-2\beta}-1 \end{array} \right) \end{array} \right] \tag{123}$$

$$kp_g^{(2)} \equiv \left[\begin{array}{c} 2\ddot{\beta}+6\frac{\dot{a}_F}{a_F}\dot{\beta}+2\dot{\beta}^2+\dot{\alpha}\dot{\beta}-\dot{\alpha}\dot{\gamma}+\dot{\beta}\dot{\gamma}+ \\ +a_F^{-2}\left(\begin{array}{c} 5e^{2(\beta-\alpha-\gamma)}-3e^{2(\gamma-\beta-\alpha)}-3e^{2(\alpha-\beta-\gamma)} \\ +6e^{-2\beta}-2e^{-2\alpha}-2e^{-2\gamma}-1 \end{array} \right) \end{array} \right] \tag{124}$$

$$kp_g^{(3)} \equiv \left[\begin{array}{c} 2\ddot{\gamma}+6\frac{\dot{a}_F}{a_F}\dot{\gamma}+2\dot{\gamma}^2-\dot{\alpha}\dot{\beta}+\dot{\alpha}\dot{\gamma}+\dot{\beta}\dot{\gamma}+ \\ +a_F^{-2}\left(\begin{array}{c} 5e^{2(\gamma-\beta-\alpha)}-3e^{2(\alpha-\gamma-\beta)}-3e^{2(\beta-\gamma-\alpha)} \\ +6e^{-2\gamma}-2e^{-2\beta}-2e^{-2\alpha}-1 \end{array} \right) \end{array} \right] \tag{125}$$

(all of which are zero when $\alpha=\beta=\gamma=0$). The Bianchi identity $T^{\nu}_{\mu,\nu}$ demands $p_g^{(1)}=p_g^{(2)}=p_g^{(3)}$ so define the averaged gravitational pressure

$$kp_g \equiv \frac{1}{3}k\left(p_g^{(1)}+p_g^{(2)}+p_g^{(3)}\right)= \tag{126}$$
$$\equiv\left[\begin{array}{c} 2\ddot{\delta}+6\frac{\dot{a}_F}{a_F}\dot{\delta}+2\left(\dot{\alpha}^2+\dot{\beta}^2+\dot{\gamma}^2\right)+ \\ +2\left(\dot{\alpha}\dot{\beta}+\dot{\alpha}\dot{\gamma}+\dot{\beta}\dot{\gamma}\right)+ \\ +a_F^{-2}\left(\begin{array}{c} -e^{2(\alpha-\beta-\gamma)}-e^{2(\beta-\alpha-\gamma)}-e^{2(\gamma-\alpha-\beta)} \\ +2e^{-2\alpha}+2e^{-2\gamma}+2e^{-2\beta}-3 \end{array}\right)\end{array}\right]$$

. Finally,

$$k\left(\epsilon_g+3p_g\right)=2\ddot{\delta}+4\frac{\dot{a}_F}{a_F}\dot{\delta}+2\left(\dot{\alpha}^2+\dot{\beta}^2+\dot{\gamma}^2\right) \tag{127}$$

[13]. Define a quasi-conformal coordinate η by $cdt \equiv a_F d\eta$; note that this fixes the relationship between t and η up to the level of the characteristic length a_i and a constant which can be set to zero. Given the impossibility of selecting a unique and objective definition for the scale factor, we do *not* define the conformal time using such a function. Define a correction term to the matter energy density q such that

$$\epsilon=\epsilon_F\left(1+q\right) \tag{128}$$

. In η-time, the Einstein equations for Bianchi IX, subtracting background terms on both sides, read:

$$\left\{\begin{array}{c} \alpha'\beta'+\alpha'\gamma'+\beta'\gamma'+2\frac{a_F'}{a_F}\delta'+ \\ +e^{-2\delta}\left[\begin{array}{c} 2\left(e^{2\alpha+2\beta}+e^{2\alpha+2\gamma}+e^{2\beta+2\gamma}\right)- \\ -e^{4\alpha}-e^{4\beta}-e^{4\gamma}-3e^{2\delta}\end{array}\right]\end{array}\right\}=a_F^2k\epsilon_F q \tag{129}$$

$$\left\{\begin{array}{c} \alpha''+\frac{a_F'}{a_F}\left(2\alpha'+\delta'\right)+\alpha'\delta'+ \\ +2e^{-2\delta}\left[e^{4\alpha}-\left(e^{2\beta}-e^{2\gamma}\right)^2-e^{2\delta}\right]\end{array}\right\}=\frac{1-w}{2}a_F^2k\epsilon_F q \tag{130}$$

$$\left\{\begin{array}{c} \beta''+\frac{a_F'}{a_F}\left(2\beta'+\delta'\right)+\beta'\delta'+ \\ +2e^{-2\delta}\left[e^{4\beta}-\left(e^{2\gamma}-e^{2\alpha}\right)^2-e^{2\delta}\right]\end{array}\right\}=\frac{1-w}{2}a_F^2k\epsilon_F q \tag{131}$$

$$\left\{\begin{array}{c} \gamma''+\frac{a_F'}{a_F}\left(2\gamma'+\delta'\right)+\gamma'\delta'+ \\ +2e^{-2\delta}\left[e^{4\gamma}-\left(e^{2\alpha}-e^{2\beta}\right)^2-e^{2\delta}\right]\end{array}\right\}=\frac{1-w}{2}a_F^2k\epsilon_F q \tag{132}$$

. We also note the Einstein equations have an exact formal solution

$$k\epsilon=\left(Sa_F^{-3}e^{-\delta}\right)^{1+w} \tag{133}$$

where S is a constant of proportionality such that S^{1+w} has dimensionality of length to the $1+3w$ power. Finally, if we define the gravitational equation of state $w_g \equiv p_g/\epsilon_g$ we see that necessarily $w_g = w$ and the Einstein equations can be read as

[13]Equation (127) corrects an error of sign in [10, equation (27)].

$$kp_g^{(1)} + wa_F^2 k\epsilon_F q = kp_g^{(2)} + wa_F^2 k\epsilon_F q = kp_g^{(3)} + wa_F^2 k\epsilon_F q = a_F^2 k\epsilon_F q + k\epsilon_g = 0 \tag{134}$$

. In other words, the effective energy-momentum tensor created by cosmological gravitational waves equals minus the back-reaction on matter energy density and pressure. Note that the quantity $k\epsilon_g/q$ is necessarily negative.

11.2 Solutions to the Einstein equations at zero order

For convenience, define the variable $x \equiv \frac{1+3w}{2}\eta$. Then at zero order the Einstein equations for a Bianchi IX universe have, for arbitrary constant equation of state, the following solution and auxiliary quantities, which are identical to the solutions to the Einstein equations in the unperturbed closed Friedmann cosmology:

$$a_F = a_i (\sin x)^{\frac{2}{1+3w}} \tag{135}$$

$$a'_F = a_i (\sin x)^{\frac{1-3w}{1+3w}} \cos x \tag{136}$$

$$a''_F = \frac{1+3w}{2} a_i \left[\frac{1-3w}{1+3w} (\sin x)^{\frac{-6w}{1+3w}} \cos^2 x - (\sin x)^{\frac{2}{1+3w}} \right] \tag{137}$$

$$a'_F/a_F = \cot x \tag{138}$$

$$H_F = a_i^{-1} \cot x \csc x \tag{139}$$

$$Q_F = \frac{1+3w}{2} \sec^2 x \tag{140}$$

. The quantity a_i represents a characteristic scale for the universe, and in the background case represents the radius of curvature of the universe at the extent of its maximum expansion. We treat a_i as an arbitrary constant for the time being.

11.3 Solutions at linear order

To first order the Einstein equations take the form:

$$2\frac{a'_F}{a_F}\delta'_1 - 2\delta_1 = a_F^2 k\epsilon_F q_1 \tag{141}$$

$$\alpha''_1 + \frac{a'_F}{a_F}(2\alpha'_1 + \delta'_1) + 8\alpha_1 - 4\delta_1 = \frac{1-w}{2} a_F^2 k\epsilon_F q_1 \tag{142}$$

$$\beta''_1 + \frac{a'_F}{a_F}(2\beta'_1 + \delta'_1) + 8\beta_1 - 4\delta_1 = \frac{1-w}{2} a_F^2 k\epsilon_F q_1 \tag{143}$$

$$\gamma''_1 + \frac{a'_F}{a_F}(2\gamma'_1 + \delta'_1) + 8\gamma_1 - 4\delta_1 = \frac{1-w}{2} a_F^2 k\epsilon_F q_1 \tag{144}$$

where the subscript 1 denotes a first-order small quantity, that is, a quantity small such that in the first approximation its square is negligible. The formal solution (133) gives us, to first order,

$$a_F^2 k\epsilon_F q_1 = -\left(1+w\right) S^{1+w} a_F^{-1-3w} \delta_1 \tag{145}$$

. Meanwhile, we can always choose to let S take on its Friedmannian value[10], so $S^{1+w} = 3a_i^{1+3w}$. Therefore:

$$\begin{aligned}
2\frac{a_F'}{a_F}\delta_1' + \left[3\left(1+w\right)\csc^2 x - 2\right]\delta_1 =& 0 && (146)\\
\alpha_1'' + 2\frac{a_F'}{a_F}\alpha_1' + 8\alpha_1 + \frac{a_F'}{a_F}\delta_1' + \left(3\frac{1-w^2}{2}\csc^2 x - 4\right)\delta_1 =& 0 && (147)\\
\beta_1'' + 2\frac{a_F'}{a_F}\beta_1' + 8\beta_1 + \frac{a_F'}{a_F}\delta_1' + \left(3\frac{1-w^2}{2}\csc^2 x - 4\right)\delta_1 =& 0 && (148)\\
\gamma_1'' + 2\frac{a_F'}{a_F}\gamma_1' + 8\gamma_1 + \frac{a_F'}{a_F}\delta_1' + \left(3\frac{1-w^2}{2}\csc^2 x - 4\right)\delta_1 =& 0 && (149)
\end{aligned}$$

which gives us the solution:

$$\delta_1 = c_1 \cos x \left(\csc x\right)^{\frac{3+3w}{1+3w}} \tag{150}$$

. The term governed by c_1 is a "removable" perturbation, that is, one not arising from a physical phenomenon but from small changes in our selection of the scale factor. Grishchuk, Doroshkevich & Iudin argue[10], and Grishchuk later proves in the case of high-frequency gravitational waves[103], that the the removable perturbation arises in the transformation from t-time to conformal time, and represents a small change in the value of η – that is, from the constant that has been implicitly set to zero in the relationship $dt = a_F d\eta$ so that $t = 0$ coincides with the metric singularity from $a_F\left(0\right) = 0$. This coincides with the argument made by Bardeen[105] with regard to scalar and vector perturbations with wavelengths longer than the Hubble radius; Bardeen recommends a gauge choice minimizing shear. We always have the freedom to set c_1 to zero but do not do so yet. In a radiation-dominated universe, we have

$$\delta_1^{\text{radiation}} = c_1^{\text{radiation}} \cos\eta \csc^2 \eta \tag{151}$$

and in a matter-dominated universe

$$\delta_1^{\text{matter}} = c_1^{\text{matter}} \cos\frac{\eta}{2} \csc^3 \frac{\eta}{2} \tag{152}$$

. Therefore the full first-order functions can be written:

$$\alpha_1'' + 2\cot x \alpha_1' + 8\alpha_1 = 3c_1 \left[\begin{array}{c} 1 + \frac{1+w}{2}\left(\csc x\right)^{1+3w} - \\ -\frac{1-w^2}{2}\csc^2 x \end{array}\right] \left(\csc x\right)^{\frac{3+3w}{1+3w}} \cos x \tag{153}$$

etc. Note that the right hand side contains *no* physical variables – no characteristic length or energy density. The Einstein equations at first order have the solutions (denoted with a tilde for the $c_1 = 0$ case)

$$\begin{aligned}\tilde{\alpha}_1^{\text{radiation}} &= (C_{\alpha 1,1}\sin 3\eta + C_{\alpha 2,1}\cos 3\eta)\csc\eta \\ \tilde{\beta}_1^{\text{radiation}} &= (C_{\beta 1,1}\sin 3\eta + C_{\beta 2,1}\cos 3\eta)\csc\eta \\ \tilde{\gamma}_1^{\text{radiation}} &= (C_{\gamma 1,1}\sin 3\eta + C_{\gamma 2,1}\cos 3\eta)\csc\eta\end{aligned} \tag{154}$$

and similarly for $\tilde{\beta},\tilde{\gamma}$ in a radiation-dominated universe, and

$$\tilde{\alpha}_1^{\text{matter}} = \frac{C_{\alpha 1,1}}{\sin\eta/2}\frac{d}{d\eta}\frac{\sin 3\eta}{\sin\eta/2} + \frac{C_{\alpha 2,1}}{\sin\eta/2}\frac{d}{d\eta}\frac{\cos 3\eta}{\sin\eta/2} \tag{155}$$

etc. in a matter-dominated universe, in both cases constrained by the condition $C_{\alpha 1,1} + C_{\beta 1,1} + C_{\gamma 1,1} = C_{\alpha 2,1} + C_{\beta 2,1} + C_{\gamma 2,1} = 0$. A general solution for any constant equation of state, in terms of orthogonal polynomials in a, exists but is far too cumbersome to be of practical use in this work. We introduce the notation $C_{\alpha 1,1}$ *etc.* to be read in the following way: $C_{\alpha 2,1}$ is an arbitrary constant associated with the function α, the first index denoting the mode of the solution (1 for growing, 2 for decaying), the second index denoting the order of the constant in an expansion assuming $\alpha,\beta,\gamma \ll 1$. For convenience, we will sometimes write a generic solution to the differential equation (153) as

$$\tilde{\alpha}_1 = C_{\alpha 1,1}y_1 + C_{\alpha 2,1}y_2 \tag{156}$$

. These solutions can be written in a less symmetric but easier-to-manipulate form:

$$\tilde{\alpha}_1^{\text{radiation}} = C_{\alpha 1,1}\left(2\cos 2\eta + 1\right) + C_{\alpha 2,1}\cos 3\eta\csc\eta \tag{157}$$

$$\tilde{\alpha}_1^{\text{matter}} = -\left[\begin{array}{c} C_{\alpha 1,1}\left(16\cos 2\eta + 10\cos\eta + 9\right) + \\ +\frac{1}{4}C_{\alpha 2,1}\csc^3\frac{\eta}{2}\left(5\cos\frac{7}{2}\eta - 7\cos\frac{5}{2}\eta\right)\end{array}\right] \tag{158}$$

etc. When $\delta = 0$ we recognize the homogeneous first-order Einstein equations as describing weak gravitational waves with wavenumber $n = 3$ and a wave equation of the form

$$\nu'' + 2\cot(x)\,\nu' + \left(n^2 - 1\right)\nu = 0 \tag{159}$$

, in line with [10]'s description. In a radiation-dominated universe we have explicitly for the full first-order solution:

$$\alpha_1^{\text{radiation}} = C_{\alpha 1,1}\frac{\sin 3\eta}{\sin\eta} + C_{\alpha 2,1}\frac{\cos 3\eta}{\sin\eta} + \frac{c_1}{3}\cos\eta\csc^2\eta \tag{160}$$

etc. and in a matter-dominated universe we have

$$\alpha_1^{\text{matter}} = \frac{C_{\alpha 1,1}}{\sin\eta/2}\frac{d}{d\eta}\frac{\sin 3\eta}{\sin\eta/2} + \frac{C_{\alpha 2,1}}{\sin\eta/2}\frac{d}{d\eta}\frac{\cos 3\eta}{\sin\eta/2} + \frac{c_1}{3}\cos\frac{\eta}{2}\csc^3\frac{\eta}{2} \tag{161}$$

. It is common to refer to the decaying "cos" mode of these gravitational waves as "singularity-destroying"[10], in that they diverge as $\eta \to 0$, which could seem at first to imply $\lim_{\eta\to 0} \gamma_{ab} \to \infty$. It is worth remembering that as the functions α, β, γ appear in the metric as exponents, that is, $\gamma_{11} = a_F^2 e^{2\alpha}$ *etc.*, decaying functions are not necessarily "singularity-destroying" for the following reasons:

- their divergence must overcome the convergence of the Friedmannian term, which in the case of weak waves will occur when $w \leq 2/3$ but not generally;
- functions of the form $e^{-x^{-y}}$ for $x < 0, y < 0$ are non-analytic near $x = 0$, that is, they are not described by convergent Taylor series in that region.

As $C_{\alpha 2,1} + C_{\beta 2,1} + C_{\gamma 2,1} = 0$, either one or two decaying terms preserve the $t = 0$ singularity when the removable perturbation is removed, in a manner analogous to that found in the Kasner universe, in the case of weak gravitational waves (although the price of this is a divergence later).

When discussing high-frequency, localized waves, it is easy to define an amplitude of the waves by (for example) normalizing a root-mean-square value over the wave's period. In the case of cosmological gravitational waves however this procedure is not possible in an absolute sense due to the diverging character of the decaying mode. Fortunately, mathematical conditions on the relation of linear-order terms to quadratic-order terms revealed at quadratic order (see SECTION 11.4) cause the term "weak" to give itself an objective meaning. If we wish to normalize the growing modes, they have the following RMS values:

$$y_1^{\mathrm{RMS}} \equiv \left[\frac{2}{(1+3w)\,\pi}\int_0^{(1+3w)\pi/2} y_1^2 d\eta\right]^{1/2} \tag{162}$$

$$y_1^{\mathrm{radiation,RMS}} = \sqrt{3} \tag{163}$$

$$y_1^{\mathrm{matter,RMS}} = \sqrt{259} \approx 16.1 \tag{164}$$

.

It is interesting to note that in matter, the decaying "cos" mode of $\alpha_1, \beta_1, \gamma_1$ has the same η-dependence as the removable perturbation; a cosmologist attempting to remove what they assume, based on an incomplete picture of the sky, to be a removable perturbation may inadvertently be suppressing evidence of a gravitational wave!

Finally, the gravitational energy-momentum tensor's (entirely removable) components read, to linear order:

$$k\epsilon_{g(1)} = 3\,(1+w)\,\frac{c_1}{a_i^2}\cos x\,(\csc x)^{\frac{9+9w}{1+3w}} \tag{165}$$

$$kp_{g(1)}^{(1)} = kp_{g(1)}^{(2)} = kp_{g(1)}^{(3)} = 3w\,(1+w)\,\frac{c_1}{a_i^2}\cos x\,(\csc x)^{\frac{9+9w}{1+3w}} \tag{166}$$

while the back-reaction of the gravitational waves at linear order gives us matter EMT components which vary from background by:

$$q_1 = -3\,(1+w)\,c_1\cos x\,(\csc x)^{\frac{5+9w}{1+3w}} \tag{167}$$

; when removable perturbations have been removed, first-order weak gravitational waves have no effect on the distribution of matter.

11.4 Solutions at quadratic order

The Einstein equations to quadratic order read:

$$2\cot x\delta_2' + \left[3(1+w)\csc^2 x - 2\right]\delta_2 = \left\{ \begin{array}{c} \left[3\csc^2 x\frac{(1+w)^2}{2} - 2\right]\delta_1^2 - \\ -\frac{1}{2}\left[\delta_1'^2 - \left(\alpha_1'^2 + \beta_1'^2 + \gamma_1'^2\right)\right] + \\ +4\left(\alpha_1^2 + \beta_1^2 + \gamma_1^2\right) \end{array} \right\} \tag{168}$$

$$\alpha_2'' + \cot x\left(2\alpha_2' + \delta_2'\right) + 8\alpha_2 - 4\delta_2 = \left[\begin{array}{c} 3\frac{1-w}{2}\csc^2 x\left(-(1+w)\delta_2 + \frac{(1+w)^2}{2}\delta_1^2\right) - \\ -\alpha_1'\delta_1' + 8\left(\beta_1 - \gamma_1\right)^2 - \\ -16\alpha_1^2 + 16\alpha_1\delta_1 - 4\delta_1^2 \end{array} \right] \tag{169}$$

$$\beta_2'' + \cot x\left(2\alpha_2' + \delta_2'\right) + 8\beta_2 - 4\delta_2 = \left[\begin{array}{c} 3\frac{1-w}{2}\csc^2 x\left(-(1+w)\delta_2 + \frac{(1+w)^2}{2}\delta_1^2\right) - \\ -\beta_1'\delta_1' + 8\left(\gamma_1 - \alpha_1\right)^2 - \\ -16\beta_1^2 + 16\beta_1\delta_1 - 4\delta_1^2 \end{array} \right] \tag{170}$$

$$\gamma_2'' + \cot x\left(2\gamma_2' + \delta_2'\right) + 8\gamma_2 - 4\delta_2 = \left[\begin{array}{c} 3\frac{1-w}{2}\csc^2 x\left(-(1+w)\delta_2 + \frac{(1+w)^2}{2}\delta_1^2\right) - \\ -\gamma_1'\delta_1' + 8\left(\alpha_1 - \beta_1\right)^2 - \\ -16\gamma_1^2 + 16\gamma_1\delta_1 - 4\delta_1^2 \end{array} \right] \tag{171}$$

. Taking the ${}^{(2)}T_0^0$ equation (168) first,

$$2\cot x\delta_2' + \left[3(1+w)\csc^2 x - 2\right]\delta_2 = \left\{ \begin{array}{c} \left[3\csc^2 x\frac{(1+w)^2}{2} - \frac{2}{3}\right]\delta_1^2 - \frac{1}{3}\delta_1'^2 + \\ +\frac{1}{2}\left(\tilde{\alpha}_1'^2 + \tilde{\beta}_1'^2 + \tilde{\gamma}_1'^2\right) + \\ +4\left(\tilde{\alpha}_1^2 + \tilde{\beta}_1^2 + \tilde{\gamma}_1^2\right) \end{array} \right\} \tag{172}$$

. The homogeneous part $\tilde{\delta}_2$ of course has the same form as at first order, representing a removable perturbation, so

$$\tilde{\delta}_2 = c_2\cos x\left(\csc x\right)^{\frac{3+3w}{1+3w}} \tag{173}$$

. The complete solution in integral form is

$$\delta_2 = \frac{1}{2}\cos x\left(\csc x\right)^{\frac{3+3w}{1+3w}} \times \tag{174}$$
$$\times\left\{ \int \left[\begin{array}{c} \left[3\csc^2 x\frac{(1+w)^2}{2} - \frac{2}{3}\right]\delta_1^2 - \\ -\frac{1}{3}\delta_1'^2 + \\ +\frac{1}{2}\left(\tilde{\alpha}_1'^2 + \tilde{\beta}_1'^2 + \tilde{\gamma}_1'^2\right) + \\ +4\left(\tilde{\alpha}_1^2 + \tilde{\beta}_1^2 + \tilde{\gamma}_1^2\right) \end{array} \right] \sec^2 x\left(\sin x\right)^{\frac{4+6w}{1+3w}}\, d\eta + c_2 \right\}$$

. Define the following pseudo-vectors and their Euclidean dot products

$$\left(C_{\alpha 1(1)}, C_{\beta 1(1)}, C_{\gamma 1(1)}\right) \equiv \boldsymbol{\sigma} \tag{175}$$
$$\left(C_{\alpha 2(1)}, C_{\beta 2(1)}, C_{\gamma 2(1)}\right) \equiv \boldsymbol{\tau} \tag{176}$$
$$\left(C^2_{\alpha 1(1)} + C^2_{\beta 1(1)} + C^2_{\gamma 1(1)}\right) = \boldsymbol{\sigma}\cdot\boldsymbol{\sigma} \qquad \equiv \sigma^2 \tag{177}$$
$$\left(C^2_{\alpha 2,1} + C^2_{\beta 2,1} + C^2_{\gamma 2,1}\right) = \boldsymbol{\tau}\cdot\boldsymbol{\tau} \qquad \equiv \tau^2 \tag{178}$$
$$\left(C_{\alpha 1,1}C_{\alpha 2,1} + C_{\beta 1,1}C_{\beta 2,1} + C_{\gamma 1,1}C_{\gamma 2,1}\right) = \boldsymbol{\sigma}\cdot\boldsymbol{\tau} \tag{179}$$

so

$$\tilde{\alpha}_1^2 + \tilde{\beta}_1^2 + \tilde{\gamma}_1^2 = \sigma^2 y_1^2 + \tau^2 y_2^2 + 2\boldsymbol{\sigma}\cdot\boldsymbol{\tau} y_1 y_2 \tag{180}$$
$$\tilde{\alpha}_1'^2 + \tilde{\beta}_1'^2 + \tilde{\gamma}_1'^2 = \sigma^2 y_1'^2 + \tau^2 y_2'^2 + 2\boldsymbol{\sigma}\cdot\boldsymbol{\tau} y_1' y_2' \tag{181}$$

. Note that the solution $\delta_2 = 0$ is excluded except in the case of a vacuum; this further excludes the definition

$$a^2 \equiv \left(\gamma_{11}\gamma_{22}\gamma_{33}\right)^{1/3} \tag{182}$$

as a useful definition of the scale factor. More careful examination reveals that the definition

$$a^2 \equiv \gamma_{ab}\eta^{ab} \tag{183}$$

is incompatible with the second-order equation as well. When all removable perturbations are set to zero,

$$\begin{aligned}\delta_2^{\text{non-removable}} =& \cos x \left(\csc x\right)^{\frac{3+3w}{1+3w}} \times \\ & \times \left\{ \int \begin{bmatrix} \sigma^2\left(2y_1^2 + \frac{1}{4}y_1'^2\right) + \\ +\tau^2\left(2y_2^2 + \frac{1}{4}y_2'^2\right) + \\ +\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\left(4y_1y_2 + \frac{1}{2}y_1'y_2'\right)\end{bmatrix} \tan^2 x \left(\sin x\right)^{\frac{2}{1+3w}} d\eta \right\}\end{aligned} \tag{184}$$

.[14] We will discuss solutions to this equation term-by-term, noting that these terms can be solved entirely from information we obtained at first order.[15]

Contributions from the removable perturbations

Contributions from the removable perturbations at second order have the explicit forms:

[14]The Einstein equations for weak gravitational waves in a Bianchi IX universe have the elegant feature of being integrable in closed form, always reducible to functions form $\sin(n\eta)\csc^k(\eta)$ and $\cos(n\eta)\csc^k(\eta)$. Theoreticians working in regimes of higher-frequency gravitational waves in a flat background may find it felicitous to approximate a Euclidean universe as a closed one in order to avoid mathematical inconveniences associated with the function $\operatorname{sinc}(t)$!

[15]Li and Schwarz[107] obtain a similar result for a flat universe, but apply their results to a different domain. The averaging scheme they propose is not an applicable approach for cosmological gravitational waves. The result is generally stated in [2, ss. 96].

In a radiation-dominated universe:

$$\delta_2^{\text{removable}} = -\frac{c_1^2}{12}\left(4\sin^2\frac{\eta}{2} + \tan^2\frac{\eta}{2} + \cot^2\frac{\eta}{2} - 2\right)\csc^2\eta + c_2\cot\eta\csc\eta \tag{185}$$

Note that the terms deriving from the first-order removable perturbation diverge as $\mathcal{O}\left(\eta^{-4}\right)$, while those from the second-order removable perturbation diverge more slowly, as $\mathcal{O}\left(\eta^{-2}\right)$.

In a matter-dominated universe:

$$\delta_2^{\text{removable}} = -\frac{c_1^2}{12}\left(3\csc^4\frac{\eta}{2} + 8\csc^2\frac{\eta}{2} - 10\right)\csc^2\frac{\eta}{2} + c_2\cot\frac{\eta}{2}\csc^2\frac{\eta}{2} \tag{186}$$

. Similarly, terms deriving from the first-order perturbation diverge as $\mathcal{O}\left(\eta^{-6}\right)$ and so at small η will dominate terms deriving from the second-order removable perturbation which diverges as $\mathcal{O}\left(\eta^{-3}\right)$.

Contributions from the growing mode

Contributions from the growing mode have the following forms:

$$\delta_2^{\text{growing}} = \sigma^2\cot x\left(\csc x\right)^{\frac{2}{1+3w}}\int\tan^2 x\left(\frac{1}{4}y_1'^2 + 2y_1^2\right)\left(\sin x\right)^{\frac{2}{1+3w}}d\eta \tag{187}$$

. We can already discern that the sign on $\delta_2^{\text{growing}}$ must be positive in a young universe.

In a radiation-dominated universe:

$$\delta_2^{\text{growing,radiation}} = \sigma^2_{\text{radiation}}\cot\eta\csc\eta\left(-\frac{1}{3}\cos 3\eta + \frac{1}{5}\cos 5\eta + 2\sec\eta\right) \tag{188}$$

; note the diverging contribution of $\mathcal{O}\left(\eta^{-2}\right)$ from growing modes.

In a matter-dominated universe:

$$\delta_2^{\text{growing,matter}} = \sigma^2_{\text{matter}}\cot\frac{\eta}{2}\csc^2\frac{\eta}{2}\left(\begin{array}{c} -\frac{6063}{4}\eta + \frac{13001}{8}\sin\eta - \frac{3237}{8}\sin 2\eta + \\ +\frac{933}{8}\sin 3\eta - 33\sin 4\eta + \\ +\frac{32}{5}\sin 5\eta + 900\tan\frac{\eta}{2} \end{array}\right) \tag{189}$$

. In contrast to the radiation-dominated case, the growing mode's contribution does not diverge in a matter-dominated universe (the term in brackets equals $0 + \mathcal{O}\left(\eta^5\right)$). Approximating to lowest orders in η,

$$\delta_2^{\text{growing,matter}} \approx \sigma^2_{\text{matter}}\left(245\eta^2 - \frac{21641}{84}\eta^4\right) \tag{190}$$

.

Contributions from the decaying mode

In a radiation-dominated universe In a radiation-dominated universe, the functions y_1 and y_2 have the property

$$y_1^2 + y_2^2 = \csc^2 \eta \tag{191}$$

while the functions y_1' and y_2' are similarly related by

$$y_1'^2 + y_2'^2 = \left(8 \sin^2 \eta + 1\right) \csc^4 \eta \tag{192}$$

. This simplifies calculations as we can readily say

$$\delta_2^{\text{decaying}} = \tau^2 \cot \eta \csc \eta \left(\frac{17}{4} \sec \eta + \frac{1}{4} \ln \tan \frac{\eta}{2} \right) - \frac{\tau^2}{\sigma^2} \delta_2^{\text{growing}} \tag{193}$$

; in a universe old enough that the diverging terms are negligible, the decaying mode intrinsically decreases the scale factor in the same way that the growing mode intrinsically increases it.

In a matter-dominated universe In a matter-dominated universe,

$$y_1^2 + y_2^2 = \csc^4 \frac{\eta}{2} \left(9 + \frac{1}{4} \cot^2 \frac{\eta}{2} \right) \tag{194}$$

and

$$y_1'^2 + y_2'^2 = \frac{1}{16} \csc^8 \frac{\eta}{2} \left(-608 \cos \eta + 140 \cos 2\eta + 477\right) \tag{195}$$

so we can state

$$\delta_2^{\text{decaying,matter}} = \tau^2 \cos \frac{\eta}{2} \csc^3 \frac{\eta}{2} \left(\begin{array}{c} 18\eta + 2450 \tan \frac{\eta}{2} - \\ -\frac{10705}{48} \cot \frac{\eta}{2} - \\ -\frac{577}{96} \cot \frac{\eta}{2} \csc^2 \frac{\eta}{2} \end{array} \right) - \frac{\tau^2}{\sigma^2} \delta_2^{\text{growing}} \tag{196}$$

. It is interesting to note that, due to the growing mode contribution's much slower contribution to change in the scale factor, the impact of the decaying mode on the dynamics of a young universe can be many orders of magnitude greater than the impact of the growing mode even when the decaying mode is several orders of magnitude weaker than the growing mode. The ratio

$$\left| \frac{\delta_2^{\text{decaying,matter}}}{\delta_2^{\text{growing,matter}}} \right| \approx \frac{\tau^2_{\text{matter}}}{\sigma^2_{\text{matter}}} \eta^{-8} \tag{197}$$

so in a matter-dominated universe with $\eta \approx 10^{-1}$ the decaying mode will have a greater impact on cosmic dynamics as long as $\tau^2_{\text{matter}} > 10^{-8}\sigma^2_{\text{matter}}$.

Contributions from the $\boldsymbol{\sigma} \cdot \boldsymbol{\tau}$ term

The contributions are described by the equation

$$\delta_2^{\text{mixed}} = \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \cos x \left(\csc x\right)^{\frac{3+3w}{1+3w}} \int \left(4y_1y_2 + \frac{1}{2}y_1'y_2'\right) \tan^2 x \left(\sin x\right)^{\frac{2}{1+3w}} d\eta \tag{198}$$

and have the following explicit forms:

Radiation-dominated universe In a radiation-dominated universe,

$$\delta_2^{\text{mixed,radiation}} = \frac{16}{15}\boldsymbol{\sigma} \cdot \boldsymbol{\tau}_{\text{radiation}} \sin\eta \cos\eta \left(3\cos 2\eta + 2\right) \tag{199}$$

.

Matter-dominated universe In a matter-dominated universe,

$$\delta_2^{\text{mixed}} = \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \cot\frac{\eta}{2} \csc^2\frac{\eta}{2} \left(\begin{array}{c} -4\cos\eta - 24\cos^2\eta - \cos 3\eta - \\ -\frac{15}{2}\cos 4\eta + 5\cos 5\eta \end{array} \right) \tag{200}$$

.

Gravitational waves at second order

Turning now to the R_a^b equations (169, 170, 171), to second order, the Einstein equations for $\epsilon - p^{(a)}$-terms read:

$$\left\{ \begin{array}{c} \alpha_2'' + 2\cot x\alpha_2' + 8\alpha_2 \\ +\frac{1}{4}\left(\alpha_1'^2 + \beta_1'^2 + \gamma_1'^2\right) \\ -3\left[\frac{w}{2}\left(1+w\right)\csc^2 x + 1\right]\delta_2 \end{array} \right\} = \left\{ \begin{array}{c} \left[\begin{array}{c} -3 + \frac{1}{16}\left(1+3w\right)^2\tan^2 x + \\ +\frac{3}{16}\left(1+w\right)\left(3w-1\right) + \\ +\left(1+w\right)^2\left(\frac{9}{16} - \frac{3}{4}w\right)\csc^2 x \end{array} \right]\delta_1^2 - \\ -\alpha_1'\delta_1' + \left(6\beta_1^2 - 16\beta_1\gamma_1 + 6\gamma_1^2\right) - \\ -18\alpha_1^2 + 16\alpha_1\delta_1 \end{array} \right\} \tag{201}$$

etc. If we suppress all removable terms, as we must for any practical observation of second-order terms, and taking into account (184), this further simplifies to

$$\alpha_2'' + 2\cot x\alpha_2' + 8\alpha_2 - 3\left[\frac{w}{2}\left(1+w\right)\csc^2 x + 1\right]\delta_2 = \left[\begin{array}{c} -26\alpha_1^2 + \\ +14\beta_1^2 + 14\gamma_1^2 - \\ -\frac{1}{4}\left(\alpha_1'^2 + \beta_1'^2 + \gamma_1'^2\right) \end{array} \right] \tag{202}$$

. Recalling the form of the gravitational waves including the removable perturbation at first order, make the simple transformation $\alpha_2 \to \tilde{\alpha}_2 + \frac{1}{3}\delta_2$ to arrive at the equations:

$$\tilde{\alpha}_2'' + 2\cot x\tilde{\alpha}_2' + 8\tilde{\alpha}_2 = 40\left[\frac{1}{3}\left(\alpha_1^2 + \beta_1^2 + \gamma_1^2\right) - \alpha_1^2\right] \tag{203}$$

etc; we recognize that linear-order gravitational waves act as a driving force on the waves at quadratic order. The solution of this equation is straightforward but tedious and we arrive at the following solutions:

In a radiation-dominated universe

$$\alpha_2^{\text{radiation}} = \left[\begin{array}{c} C_{\alpha 1,2}\frac{\sin 3\eta}{\sin\eta} + C_{\alpha 2,2}\frac{\cos 3\eta}{\sin\eta} + \\ +40\left(\frac{1}{3}\sigma^2 - C_{\alpha 1,1}^2\right)\left(\frac{1}{36}\frac{\sin 3\eta}{\sin\eta} - \frac{1}{6}\eta\frac{\cos 3\eta}{\sin\eta}\right) + \\ +40\left(\frac{1}{3}\tau^2 - C_{\alpha 2,1}^2\right)\left(\begin{array}{c} \frac{1}{6}\eta\frac{\cos 3\eta}{\sin\eta} + \frac{1}{36}\frac{\sin 3\eta}{\sin\eta} + \frac{5}{24} + \\ +\frac{1}{16}\frac{\sin 5\eta}{\sin\eta} - \frac{1}{6}\frac{(2\eta-\pi)\cos 3\eta - 2\sin 3\eta\ln(2\sin\eta)}{\sin\eta} \end{array}\right) + \\ +40\left(\frac{2}{3}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - 2C_{\alpha 1,1}C_{\alpha 2,1}\right)\left(\begin{array}{c} \frac{1}{6}\eta\frac{\sin 3\eta}{\sin\eta} + \frac{1}{8}\cot\eta + \\ +\frac{1}{36}\frac{\cos 3\eta}{\sin\eta} - \frac{1}{32}\frac{\cos 5\eta}{\sin\eta} \end{array}\right) \end{array}\right] + \frac{1}{3}\delta_2 \tag{204}$$

etc. with the second-order constants $C_{\alpha 1,2}$ *etc.* constrained such that

$$C_{\alpha 1,2} + C_{\beta 1,2} + C_{\gamma 1,2} = C_{\alpha 1,2} + C_{\beta 1,2} + C_{\gamma 1,2} = 0 \tag{205}$$

. To lowest order in η the solution for α_2 reads

$$\alpha_2^{\text{radiation}} \approx \left[\begin{array}{c} C_{\alpha 1,2}\left(3 - 4\eta^2\right) + 20\left(\frac{1}{3}\sigma^2 - C_{\alpha 1,1}^2\right)\left(-\frac{1}{6} + \frac{11}{9}\eta^2\right) \\ +C_{\alpha 2,2}\eta^{-1} + \frac{20\pi}{3}\left(\frac{1}{3}\tau^2 - C_{\alpha 2,1}\right)\eta^{-1} + \\ +\frac{175}{36}\left(\frac{2}{3}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - 2C_{\alpha 1,1}C_{\alpha 2,1}\right)\eta^{-1} + \frac{1}{3}\delta_2^{\text{non-removable}} \end{array}\right] \tag{206}$$

etc. For the pure decaying mode, the contribution from δ_2 dominates, while for the pure growing mode and the mixed term the contributions from the homogeneous parts of α_2 dominate.

In a matter-dominated universe For a matter-dominated universe, the gravitational wave equation to second order has the following solution[16]:

$$\alpha_2 = \left(\begin{array}{c} C_{\alpha 1,2}\csc\frac{\eta}{2}\frac{d}{d\eta}\frac{\sin 3\eta}{\sin\eta/2} + C_{\alpha 2,2}\csc\frac{\eta}{2}\frac{d}{d\eta}\frac{\cos 3\eta}{\sin\eta/2} + \\ +\alpha_2^{\text{growing}} + \alpha_2^{\text{decaying}} + \alpha_2^{\text{mixed}} + \frac{1}{3}\delta_2^{\text{non-removable}} \end{array}\right) \tag{207}$$

[16] There is no "royal road" to the explicit statement of this function, which was derived by substitution and variation of parameters with the assistance of a computer algebra system. With foreknowledge of the form of the solution, the equation (203) can be solved through the method of undetermined coefficients; this requires solving a 21-dimensional linear system. (203) may also admit a solution through the method of Fourier transforms, but only under torture.

$$\alpha_2^{\text{growing}} \equiv 5\left(\frac{1}{3}\sigma^2 - C_{\alpha 1,1}^2\right)\left\{\begin{array}{c} \frac{1}{70}\sum_{n=0}^{10} g_n \cos n\eta + \\ +\frac{1}{56}\csc^3\frac{\eta}{2}\left[\begin{array}{c} \eta\left(\begin{array}{c} -1128960\cos\frac{5\eta}{2} + \\ +806400\cos\frac{7\eta}{2} \end{array}\right) + \\ +\sum_{n=0}^{11} h_n \sin\left(\frac{2n+1}{2}\eta\right) \end{array}\right] \end{array}\right\} \tag{208}$$

$$\begin{aligned} g_0 &= 32900, g_1 = 443310, g_2 = 90230, g_3 = 354221, g_4 = 20195, g_5 = 248918, \\ &g_6 = -57025, g_7 = 68911, g_8 = -37880, g_9 = 15440, g_{10} = -22400 \\ h_0 &= 1166543, h_1 = -1664285, h_2 = 888216, h_3 = 990580, h_4 = -1262310, h_5 = 677390, \\ &h_6 = -363895, h_7 = 197841, h_8 = -116900, h_9 = 66864, h_{10} = -34304, h_{11} = 8960 \end{aligned}$$

$$\alpha_2^{\text{growing}} \approx \left(\frac{1}{3}\sigma^2 - C_{\alpha 1,1}^2\right)\left(82630 - \frac{4513087}{7}\eta^2\right) \tag{209}$$

$$\alpha_2^{\text{decaying}} \equiv \frac{1}{245}\left(\frac{1}{3}\tau^2 - C_{\alpha 2,1}^2\right)\csc^4\frac{\eta}{2}\left(\begin{array}{c} -\frac{\eta}{2}\tan\frac{\eta}{2}\sum_{n=0}^{4} j_n \cos^n\eta + \\ +\sum_{n=0}^{6} k_n \cos^n\eta + \\ +\ln\left(-2\sin^2\frac{\eta}{2}\right)\sum_{n=0}^{4} l_n \cos^n\eta \end{array}\right) \tag{210}$$

$$\begin{aligned} j_0 &= -34020, j_1 = -17010, j_2 = 153090, j_3 = 22680, j_4 = -113400 \\ &k_0 = 58329, k_1 = -514422, k_2 = 368937, k_3 = 675396, \\ &k_4 = -678540, k_5 = 31500, k_6 = 61250 \\ &l_0 = -5670, l_1 = 102060, l_2 = -73710, l_3 = -136080, l_4 = 113400 \end{aligned}$$

$$\alpha_2^{\text{mixed}} \equiv \frac{4}{105}\left(\frac{1}{3}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - C_{\alpha 1,1}C_{\alpha 2,1}\right)\csc^2\frac{\eta}{2}\left(\begin{array}{c} \frac{\eta}{2}\sum_{n=0}^{3} m_n \cos\eta - \\ -\cot\frac{\eta}{2}\sum_{n=0}^{5} n_n \cos^n\eta \end{array}\right) \tag{211}$$

$$\begin{aligned} m_0 &= 2310, m_1 = -39270, m_2 = -9240, m_3 = 46200 \\ &n_0 = -936, n_1 = 15693, n_2 = 30204, n_3 = -58700, n_4 = -25200, n_5 = 42000 \end{aligned}$$

$$\alpha_2^{\text{mixed}} \approx -\frac{32}{105}\left(\frac{1}{3}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} - C_{\alpha 1,1}C_{\alpha 2,1}\right)\eta^{-3}\sum_{n=0}^{5} n_n$$

etc. The statement of the solutions to the gravitational wave equations to quadratic order in the matter-dominated universe are original to this work; the radiation-dominated quadratic order wave equations were presented in [10]. Note that $\sum_n l_n = \sum_n m_n = 0$.

Most interesting is the presence of ln-terms in (193) and (210), which on the one hand indicate the appearance of the power-law behavior of metric coefficients which typify the Kasner universe and the BKL universe in its quasi-isotropic phase; but which on the other hand show the breakdown

of our approximation scheme and the limit of regular perturbation theory in solving the problem to hand; the Taylor expansion of the growing mode of α_2 indicates further that waves must be very weak ($\|\sigma\| = \mathcal{O}\left(10^{-4}\right)$) for the approximation scheme to be rigorously valid. In any case, indications are that the growing mode of hypothetical cosmological gravitational waves should be very much stronger than the decaying mode (see SECTION 19); we will not need to make use of the second-order solutions for the decaying mode and from here on will treat the decaying mode as being linear-order weak, that is, $C^2_{\alpha 2,1} \approx C^2_{\beta 2,1} \approx C^2_{\gamma 2,1} \approx C_{\alpha 2,2} \approx C_{\beta 2,2} \approx C_{\gamma 2,2} \approx \tau^2 \approx 0$.

11.5 Strong growing waves in the quasi-isotropic regime

[10, part 3] begins the development of equations for a radiation-dominated universe describing strong gravitational waves in Bianchi IX. Similar equations in a matter-dominated universe are useful in considering observed acceleration, as $\Delta Q \approx -1$.

Consider the equations (129-132). Assume a solution of the form

$$\begin{aligned}
\alpha &= \sum_{n=0}^{\infty} c^{\alpha}_{2n} \eta^{2n} \\
\beta &= \sum_{n=0}^{\infty} c^{\beta}_{2n} \eta^{2n} \\
\gamma &= \sum_{n=0}^{\infty} c^{\gamma}_{2n} \eta^{2n}
\end{aligned} \tag{212}$$

with the terms c^{ξ}_{n} constants. It is convenient to define $e^{2c^{\alpha}_0} \equiv A, e^{2c^{\beta}_0} \equiv B, e^{2c^{\gamma}_0} \equiv G$. In a matter-dominated universe, to lowest two orders the solutions read

$$\begin{aligned}
\alpha \approx& c^{\alpha}_0 + \frac{1}{20}\left[1 - \frac{1}{ABG}\left(5A^2 - 3B^2 - 3G^2 + 6BG - 2AB - 2AG\right)\right]\eta^2 \\
\beta \approx& c^{\beta}_0 + \frac{1}{20}\left[1 - \frac{1}{ABG}\left(5B^2 - 3G^2 - 3A^2 + 6AG - 2BG - 2AB\right)\right]\eta^2 \\
\gamma \approx& c^{\gamma}_0 + \frac{1}{20}\left[1 - \frac{1}{ABG}\left(5G^2 - 3A^2 - 3B^2 + 6AB - 2AG - 2BG\right)\right]\eta^2
\end{aligned} \tag{213}$$

where $c^{\alpha}_0, c^{\beta}_0, c^{\gamma}_0$ are arbitrary; if we want to preserve the Friedmannian value of S then we need

$$c^{\alpha}_0 + c^{\beta}_0 + c^{\gamma}_0 = 0 \tag{214}$$

[10]. We always have the freedom to set one of these to zero by a simple scaling of the metric; this preserves the two degrees of freedom for the gravitational wave.

If we apply the condition (214) and set the parameter $c^{\gamma}_0 = 0$ by scaling, then the strong growing-mode waves are described by

$$\begin{aligned} c_0^\alpha &\in \mathbb{R} && (215)\\ c_0^\beta &= -c_0^\alpha && (216)\\ c_0^\gamma &= 0 && (217)\\ c_2^\alpha &= \frac{1}{20}\left(-5A^2+2A+6-6A^{-1}+3A^{-2}\right) && (218)\\ c_2^\beta &= \frac{1}{20}\left(3A^2-6A+6+2A^{-1}-5A^{-2}\right) && (219)\\ c_2^\gamma &= \frac{1}{20}\left(3A^2+2A-10+2A^{-1}+3A^{-2}\right) && (220)\end{aligned}$$

with the single parameter c_0^α determining the whole system. Note that setting $c_0^\gamma = 0$ does not imply $\gamma' = 0$. We can also qualitatively say that for any value of A, two of functions α, β, γ will be positive, as will δ, unless $A = 1$ (the background case), in the regime that $A\eta$ is sufficiently small that $A^3\eta^3$ is negligible.

The functions (212) are linearly independent with y_2^{matter} to lowest order in η and therefore can be used together to describe a matter-dominated universe with arbitrarily strong growing gravitational waves and weak decaying gravitational waves up to order η^2, as long as the series (212) converge.

11.6 Dynamics

As in the Kasner universe (see SECTION 9.1), it is useful to generalize quantities pertaining to the expansion of space which are spherically symmetric in Friedmannian cosmology.

In terms of our statement of the metric (105), the generalized dynamical quantities for our space are

$$a_{ab} = a_F \begin{pmatrix} e^\alpha & 0 & 0 \\ 0 & e^\beta & 0 \\ 0 & 0 & e^\gamma \end{pmatrix} \tag{221}$$

$$\bar{a} = \frac{1}{3} a_F \left(e^\alpha + e^\beta + e^\gamma\right) \tag{222}$$

$$H_{ab} = \begin{pmatrix} \dot{a}_F/a_F + \dot{\alpha} & 0 & 0 \\ 0 & \dot{a}_F/a_F + \dot{\beta} & 0 \\ 0 & 0 & \dot{a}_F/a_F + \dot{\gamma} \end{pmatrix} \tag{223}$$

$$\bar{H} = \frac{\dot{a}_F}{a_F} + \frac{1}{3}\dot{\delta} \tag{224}$$

$$Q^1_1 \equiv \frac{d}{dt} H^{1c}\eta_{1c} - \delta^1_1 = -\frac{\begin{pmatrix} \ddot{a}_F/a_F + 2H_F\dot{\alpha} + \\ +\ddot{\alpha} + \dot{\alpha}^2 \end{pmatrix}}{\left(H_F + \dot{\alpha}\right)^2} \tag{225}$$

etc.

$$\bar{Q} = -\frac{1}{3}\left(\begin{array}{c} \frac{\ddot{a}_F/a_F+2H_F\dot{\alpha}+\ddot{\alpha}+\dot{\alpha}^2}{(H_F+\dot{\alpha})^2}+ \\ +\frac{\ddot{a}_F/a_F+2H_F\dot{\beta}+\ddot{\beta}+\dot{\beta}^2}{\left(H_F+\dot{\beta}\right)^2}+ \\ +\frac{\ddot{a}_F/a_F+2H_F\dot{\gamma}+\ddot{\gamma}+\dot{\gamma}^2}{(H_F+\dot{\gamma})^2} \end{array}\right) \tag{226}$$

. Our goal in undertaking the arduous task of solving the Einstein equations has been to derive the impact of long-wavelength gravitational waves on cosmic dynamics, particularly acceleration. We are now in a position to begin to discuss this impact.

Let each quantity in section (11.6) be expanded out into a background term plus corrections, such that for example

$$a_{ab} \approx a_{ab}^{(0)} + a_{ab}^{(1)} + a_{ab}^{(2)} \tag{227}$$

. Then the zero-order, background terms are simply

$$a_{ab}^{(0)} = a_F \eta_{ab} \tag{228}$$

$$H_{ab}^{(0)} = H_F \eta_{ab} \tag{229}$$

$${}^{(0)}Q_a^b = Q_F \delta_a^b \tag{230}$$

. While the gravitational energy-momentum tensor vanishes at first order with the removal of removable perturbations, the presence of weak gravitational waves can affect observed dynamic quantities. At first order:

$$a_{ab}^{(1)} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & \gamma_1 \end{pmatrix} \tag{231}$$

$$\bar{a}_{(1)} = \frac{1}{3}\delta_1 \tag{232}$$

$$H_{ab}^{(1)} = \begin{pmatrix} \dot{\alpha}_1 & 0 & 0 \\ 0 & \dot{\beta}_1 & 0 \\ 0 & 0 & \dot{\gamma}_1 \end{pmatrix} \tag{233}$$

$$\bar{H}_{(1)} = \frac{1}{3}\dot{\delta}_1 \tag{234}$$

$${}^{(1)}Q_1^1 = -H_F^{-1}\left[2\left(Q_F+1\right)\dot{\alpha}_1 + H_F^{-1}\ddot{\alpha}_1\right] \tag{235}$$

etc.,

$$\bar{Q}_{(1)} = -\frac{1}{3}H_F^{-1}\left[2\left(Q_F+1\right)\dot{\delta}_1 + H_F^{-1}\ddot{\delta}_1\right] \tag{236}$$

Thus we illustrate the need for truly representative sky coverage in considering the problem of acceleration: gravitational waves can contribute to anisotropic acceleration even when they do not affect the distribution of matter. In domains when the first derivatives of a wave is small (that is, near peaks and troughs of the wave), the accelerative effect will not be accompanied by a large change in the Hubble flow. As before, a failure to completely suppress the removable perturbation may lead to incorrect evaluation of the strength of decaying modes. To first order, non-zero contribution to the average over the whole sky of the perturbations is removable; first-order weak gravitational waves in Bianchi IX do not produce isotropic acceleration.

To quadratic order, the dynamic quantities have the forms

$$a_{ab}^{(2)} = a_F \begin{pmatrix} \alpha_2+\alpha_1^2/2 & 0 & 0 \\ 0 & \beta_2+\beta_1^2/2 & 0 \\ 0 & 0 & \gamma_2+\gamma_1^2/2 \end{pmatrix} \tag{237}$$

$$\bar{a}_2 = \frac{1}{3}a_F\left[\delta_2 + \frac{1}{2}\left(\alpha_1^2+\beta_1^2+\gamma_1^2\right)\right] \tag{238}$$

$$H_{ab}^{(2)} = \begin{pmatrix} \dot{\alpha}_2 & 0 & 0 \\ 0 & \dot{\beta}_2 & 0 \\ 0 & 0 & \dot{\gamma} \end{pmatrix} \tag{239}$$

$$\bar{H}_{(2)} = \frac{1}{3}\dot{\delta}_2 \tag{240}$$

$$^{(2)}Q_1^1 = -H_F^{-1}\left[\begin{array}{c} 2\left(Q_F+1\right)\dot{\alpha}_2 + H_F^{-1}\ddot{\alpha}_2 - \\ -3H_F^{-1}\left(Q_F+1\right)\dot{\alpha}_1^2 - 2H_F^{-2}\dot{\alpha}_1\ddot{\alpha}_1 \end{array}\right] \tag{241}$$

etc.,

$$\bar{Q}_{(2)} = -\frac{1}{3}H_F^{-1}\left[\begin{array}{c} 2\left(Q_F+1\right)\dot{\delta}_2 + H_F^{-1}\ddot{\delta}_2 - \\ -3H_F^{-1}\left(Q_F+1\right)\left(\dot{\alpha}_1^2+\dot{\beta}_1^2+\dot{\gamma}_1^2\right) - \\ -2H_F^{-2}\left(\dot{\alpha}_1\ddot{\alpha}_1+\dot{\beta}_1\ddot{\beta}_1+\dot{\gamma}_1\ddot{\gamma}_1\right) \end{array}\right] \tag{242}$$

. At second order we begin to see a consequence of the non-linearity of the Bianchi IX Einstein equations which is potentially very important in the study of cosmic dynamics: isotropic changes to the Hubble parameter and to acceleration from anisotropic metric terms. With our knowledge of the Einstein equations at first and second order (153,172,202) we can show this explicitly:

$$^{(2)}Q_1^1 = -\tan x\left\{\begin{array}{c} \left(3w+(1+3w)\tan^2 x\right)\alpha_2' - 8\tan x\alpha_2 - \\ -40\tan x\alpha_1^2 + \frac{3}{2}\left(1-3w-(1+3w)\tan^2 x\right)\tan x\alpha_1'^2 + \\ +16\tan^2 x\alpha_1'\alpha_1 + \\ +\tan x\left[\begin{array}{c} 3\left[\frac{w}{2}\left(1+w\right)\csc^2 x+1\right]\delta_2 + \\ +14\left(\alpha_1^2+\beta_1^2+\gamma_1^2\right) - \frac{1}{4}\left(\alpha_1'^2+\beta_1'^2+\gamma_1'^2\right) \end{array}\right] \end{array}\right\} \tag{243}$$

etc. and

$$\bar{Q}_{(2)} = \frac{1}{3}\tan^2 x \left\{ \begin{array}{c} \frac{1}{2}(1+3w)^2 \sec^2 x\delta_2 - \\ -2(1+3w)\sec^2 x\left(\alpha_1^2+\beta_1^2+\gamma_1^2\right)+ \\ +\frac{1}{4}\left[1+15w+5(1+3w)\tan^2 x\right]\left(\alpha_1'^2+\beta_1'^2+\gamma_1'^2\right)- \\ -16\tan x\left(\alpha_1'\alpha_1+\beta_1'\beta_1+\gamma_1'\gamma_1\right) \end{array} \right\} \tag{244}$$

. Isotropic acceleration with quadratic-order strength arises from the non-linear interaction of linear-order gravitational waves; but in the regime of $|\alpha|, |\beta|, |\gamma| \ll 1$ the gravitational waves at linear order will dominate measurement of cosmological parameters.

In a matter-dominated universe with η small the deceleration terms become, defining

$$\Delta Q^a_b \equiv Q^a_b - Q_F\delta^a_b \tag{245}$$

$$\Delta\bar{Q} \equiv \bar{Q} - Q_F \tag{246}$$

$$\Delta Q^1_{1,\text{matter}} \approx -\frac{\eta^2}{4}\left\{ \begin{array}{c} \tan\frac{\eta}{2}\left(\alpha_1'+\alpha_2'\right)-8\left(\alpha_1+\alpha_2\right)- \\ -40\alpha_1^2+\frac{3}{2}\alpha_1'^2+16\tan\frac{\eta}{2}\alpha_1'\alpha_1+ \\ +3\delta_2+14\left(\alpha_1^2+\beta_1^2+\gamma_1^2\right)-\frac{1}{4}\left(\alpha_1'^2+\beta_1'^2+\gamma_1'^2\right) \end{array} \right\} \tag{247}$$

$$\Delta\bar{Q}^{\text{matter}} \approx \frac{1}{48}\eta^2 \left[\begin{array}{c} 2\delta_2-8\left(\alpha_1^2+\beta_1^2+\gamma_1^2\right)+ \\ +\left(\alpha_1'^2+\beta_1'^2+\gamma_1'^2\right)- \\ -16\tan\frac{\eta}{2}\left(\alpha_1'\alpha_1+\beta_1'\beta_1+\gamma_1'\gamma_1\right) \end{array} \right] \tag{248}$$

etc. Explicitly, these will have the lowest-order forms:

$$\Delta Q^1_{1,\text{matter}} \approx -\frac{\eta^2}{4}\left(\begin{array}{c} C_{\alpha 1,1}\left(280-259\eta^2\right)+ \\ +C_{\alpha 2,1}\left(16\eta^{-3}-72\eta^{-1}+\frac{2251}{5}\eta\right)+ \\ +C^2_{\alpha 1,1}\left(710040-4687753\eta^2\right)+ \\ +\sigma^2\left(-609590/3+\frac{4402405}{3}\eta^2\right) \end{array} \right) \tag{249}$$

$$\Delta\bar{Q}^{\text{matter}} \approx \frac{\sigma^2}{24}\eta^2\left(-4900+2983\eta^2\right) \tag{250}$$

. These results are encouraging as if we choose $\|\sigma\| \sim 10^{-4}$ (in order to make the gravitational waves weak) and $\eta \sim 10^{-2}$to match (17), we obtain $\Delta Q^1_{1,\text{matter}} \sim -10^{-6}$, which has the right sign as well as all the contributions at both first and second orders going in the "right" direction, toward acceleration. It is particularly encouraging that both growing and decaying modes contribute to acceleration to their lowest orders in η.

11.7 Back-reaction

Of interest in discussing the problem of acceleration is the effective equation of state of the gravitational waves' contribution to the energy density. Empirically, the equation of state of dark energy seems to be close to $w_X = -1$ (see SECTION 6), where the quantity w_x is related to the source of the energy such that the source evolves with regard to the scale factor at a rate of $a^{-3(1+w_X)}$. As noted in (SECTION 9.1) there is no unique way to define the scale factor; but a condition of quasi-isotropy is that expansion in every direction in the current epoch is proportional, that is, that they evolve as the same power of time. If the decaying mode of the cosmological gravitational wave is weak, then this evolution will be proportional to the Friedmannian scale factor.

To quadratic order, (119) reads

$$k\epsilon_g^{(2)} = 3\,(1+w)\,a_F^{-2}\csc^2 x\delta_2 \tag{251}$$

and so by (134)

$$q_{(2)} = -\,(1+w)\,a_F^{-2}\delta_2 \tag{252}$$

. When the growing mode is dominant, δ_2 is always positive in a matter-dominated universe, and therefore $q_{(2)}$ is negative and thus the back-reaction appears to have negative energy density; a significant "mixed" $\boldsymbol{\sigma}\cdot\boldsymbol{\tau}$ term, however, can easily introduce intervals where $q_{(2)} > 0$.

In a matter-dominated universe and when the growing mode is dominant, $q_{(2)} \propto \eta^{-2}$ which, if the universe is evolving with a scale as $a_F \propto \eta^2$, implies an equation of state for the back-reaction of $w_X = -1/3$ (as compared to an equation of state for a cosmological constant of $w_X = -1$). While no investigation of the equation of state of dark energy includes this value within its highest confidence interval, measurements of w_X remain tentative, with large errors and high sensitivity both to single data points and to the algorithm for curve-fitting models to the data (see SECTION 6).

The dominant term in (119) is the a'_F/a_F-term. This stands in stark contrast to the commonly-considered case of gravitational waves in a background so slowly moving compared to the period of the waves that $\dot{a}_F \approx 0$, in which instance the quadratic combination of first-derivative terms dominates.

In regimes of stronger growing-mode gravitational waves, though, the scale factor as defined in (39) will be more dominated by terms of higher, even order and so $a_{ab} \propto \eta^4$ or higher. As the growing mode increases in strength, the equation of state decreases asymptotically toward a limit of $w_X = -1$; if the scale factor grows as η^{2s}, the equation of state for the back-reaction is given by $w_X = (1/3s) - 1$. As acceleration is empirically $Q_0 = -0.6$, this implies that in real life the gravitational wave strength is of order unity and therefore the effective equation of state is close to -1. Thus, the quasi-isotropic Bianchi IX model with strong growing-mode gravitational waves and weak or zero decaying-mode waves is compatible with the observed data on the equation of state of dark energy, without the invocation of a cosmological constant; the theory would be invalidated by definitive measurements of $w_X < -1$.

In any case, the fact of $w_X < 0$ allows us to draw a conclusion regarding cosmic evolution. [76] notes Kasner-like cosmologies go through two stages of evolution:

1. A “vacuum” stage, where matter’s influence is, due to its evolution as a^{-4}, weak compared to the influence of the anisotropic expansion and contraction, influence which, in light of (SECTION 10), we now understand to be the result of gravitational waves in the BKL universe;

2. a “matter” stage, where expansion isotropizes[106] and is driven by, first relativistic ($w = 1/3$), then cold, non-relativistic ($w = 0$) matter. Formally, the contribution of curvature to cosmic evolution becomes important in this era ($w_K = -1/3$), but as the influence of curvature will be isotropic in Bianchi IX and the radius of curvature is very large compared to the Hubble radius (see SECTION 17), curvature will not have a practical influence on observations in and of itself.[17] To this second stage we can add a third stage:

3. A “dark energy” stage, in which growing modes of the cosmological gravitational waves which drove the initial isotropy return as the dominant influence on cosmic evolution.

11.8 Amplification of gravitational waves

Grishchuk observed[8] that when the background of a cosmology containing gravitational waves varies rapidly, weak gravitational waves can be amplified where they would otherwise, in a slowly-moving background, decay rapidly[79]. With regard to the Bianchi IX cosmology this is significant as when the growing mode of a cosmological gravitational wave dominates, the leading term in the gravitational energy density is of the form $(a'_F/a_F)\,\delta'_2 = \mathcal{O}\,(\text{constant})$. Cosmological observations (see SECTION 17) indicate the universe has $\eta < \mathcal{O}\left(10^{-1}\right)$. In this regime, the term $a'_F/a_F = \cot(\eta/2) \approx 2/\eta$, which is dependent on the rate of change of the background, is arbitrarily large; therefore, weak waves may have an effect orders of magnitude greater than their amplitude. Similarly, the decaying mode of gravitational waves can have prominent or even dominant power in a sufficiently young universe even when the amplitude of the decaying mode is smaller than that of the growing mode.

12 Conclusions

Solutions have been presented for the gravitational wave equation for a Bianchi IX universe perturbed to quadratic order from the closed Friedmann case. Quadratic order is the limit of perturbation theory’s applicability to explore nearly-Friedmannian Bianchi IX when decaying modes are sufficiently strong that they are not negligible.

At quadratic order, the non-linear interaction of the gravitational waves produces isotropic changes to dynamic quantities. While this isotropic change is likely to be dominated in any particular direction by linear-order contributions from the gravitational waves, in the regime of strong gravitational waves they will become more important and potentially even dominant. Where [98] discussed the possibility of acceleration in a non-vacuum Bianchi IX universe only qualitatively, we have shown it explicitly as well as illustrating a clear link between acceleration and the gravitational waves which are intrinsic to Bianchi IX in its full generality.

[17] Formally we can also say that, due to the action of proton decay and positron annihilation, after sufficient time the $w = 0$ phase will return to a $w = 1/3$ phase where the universe is filled with neutrinos and photons. Following this period there will be another return to $w = 0$ as these free particles are absorbed by black holes. As these black holes evaporate by the process of Hawking radiation, there will then be a final return to $w = 1/3$. [80] gives a popular-science presentation of the universe in these phases, but as it was written only shortly after the discovery of acceleration its treatment of dark energy is highly speculative.

It is curious to note that the order-η^2 approximation we have made in (SECTION 11.5), α and δ in the normalization we have chosen take the form of Alexander polynomials[109, 110], although not Alexander polynomials for any knot of fewer than 11 crossings. Whether this mathematical observation is significant or coincidental is a subject for further debate, but as gravitational waves in Bianchi IX are moving equatorially around our background 3-sphere[10], and as a sub-class of knots (the "torus knots") are constructed by wrapping one 2-torus around another it is conceivable there could be a connection.

Back-reaction from growing modes of the gravitational waves appears to have negative energy density and an equation of state compatible with that observed for dark energy, especially in the regime of strong gravitational waves and quasi-isotropic expansion; when gravitational waves are strong, they become the dominant contributor to the evolution of the cosmos in an era following the era of matter domination.

Therefore, from the perspective of cosmic dynamics, cosmological gravitational waves in a quasi-isotropic Bianchi IX universe are a viable candidate for dark energy, without the invocation of a cosmological constant and without requiring any modification of the theory of relativity. An analysis of the impact of these gravitational waves on the cosmic microwave background is necessary in order to determine whether constraints from the CMB are compatible with the observed data on acceleration.

Part IV

The Cosmic Microwave Background of a Bianchi IX universe

While long-wavelength gravitational waves can cause acceleration in a Bianchi IX universe, the effect of such waves must be compatible with the observed cosmic microwave background in order to represent a practical model for explaining observed acceleration.

Sachs & Wolfe initiated[23] the systematic study of the effect of perturbations on the CMB, following a formalism developed by Kristian & Sachs[25]. Sachs & Wolfe's work developed the theory of scalar, vector and tensor perturbations on the CMB in a flat almost-isotropic universe to first order.

Sachs & Wolfe's work was generalized by Anile & Motta[26] to the almost-isotropic closed and open Friedmann cosmologies, again at first order. While Anile & Motta begin to consider the impact of long-wavelength gravitational waves on the CMB, they choose to explore the impact of waves with scales much smaller than the Hubble radius. Anile & Motta subsequently[27] ruled out the existence of these waves at significant strengths in the observable universe.

Doroshkevitch, Lukash & Novikov considered the impact of an anisotropic universe on the CMB in the case of the Bianchi VII, VIII and IX models[19], and concluded that a Bianchi IX model was potentially "compatible with observations, only if there was some secondary heating of the intergalactic gas". Doroshkevitch *et al*'s most important calculations are carried out on the assumption, then widespread, of $\Omega_M \approx 1$ and as such are of limited applicability; interestingly, though, in their conclusions they note that if $\Omega_M < 1$, "$\Delta T/T$ will be close to the maximum value only in a small 'spot' with an angular size $\theta \approx 4\Omega$" (where by "small" they give the example

of $\Omega_M \approx 0.1 \implies \theta \approx 23°$).

Sung & Coles analytically and computationally explore the impact of various unperturbed Bianchi models, including Bianchi IX, on the CMB[21]. They report the useful theorem that "a gravitational field alone is not able to generate polarization", but do not consider the general case of Bianchi IX, only the isotropic case equivalent to the closed Friedmann universe.

13 Geodesic equations

The effect of the metric on the CMB is determined by examining the change in geodesics of light rays relative to an isotropic, background case. Let the subscript E denote a function evaluated at the time of the emission of a photon, and the subscript R denote that function evaluated at the time of the photon's reception. Then the change in the temperature of the background radiation T is given by

$$T_R/T_E = \frac{1}{z+1} \tag{253}$$

. Consider the path of a light ray; let this be a four-vector denoted by k^μ such that $k^\mu k_\mu = 0$, with the light ray received in the direction $k^i_R = e^i$. The geodesic equation for the time part of k^μ in a Bianchi cosmology reads

$$\frac{dk^0}{d\lambda} + \Gamma^0_{ij} k^i k^j = 0 \tag{254}$$

and the equations for the space part of the vector read

$$\frac{dk^a}{d\lambda} + \Gamma^a_{00} + \Gamma^a_{0i} k^i + \Gamma^a_{i0} k^i + \Gamma^a_{bc} k^b k^c = 0 \tag{255}$$

Recalling (82) and (105) the Christoffel symbols

$$\Gamma^0_{ij} = \frac{1}{2} \gamma_{ab,0} e^a_i e^b_j \tag{256}$$

, $\Gamma^a_{00} = \Gamma^a_{0i} = \Gamma^a_{i0} = 0$ and the Ricci rotation coefficients read[18]:

[18]The symbol ε_{abc} represents the Levi-Civita symbol defined such that $\varepsilon_{123} = 1$

$$
\begin{aligned}
\Gamma^a_{bc} =& \frac{1}{2}\left(\delta^a_f \epsilon_{bcd} + \gamma^{ag}\gamma_{cd}\epsilon_{gbf} - \gamma^{ag}\gamma_{db}\epsilon_{cgf}\right)\eta^{df} && \\
\Gamma^1_{23} =& \frac{1}{2}\left(\gamma^{11}\left(\gamma_{33}-\gamma_{22}\right)+1\right) = && \frac{1}{2}\left(e^{2\gamma-2\alpha} - e^{2\beta-2\alpha}+1\right) \\
\Gamma^1_{32} =& \frac{1}{2}\left(\gamma^{11}\left(\gamma_{33}-\gamma_{22}\right)-1\right) = && \frac{1}{2}\left(e^{2\gamma-2\alpha} - e^{2\beta-2\alpha}-1\right) \\
\Gamma^2_{31} =& \frac{1}{2}\left(\gamma^{22}\left(\gamma_{11}-\gamma_{33}\right)+1\right) = && \frac{1}{2}\left(e^{2\alpha-2\beta} - e^{2\gamma-2\beta}+1\right) \\
\Gamma^2_{13} =& \frac{1}{2}\left(\gamma^{22}\left(\gamma_{11}-\gamma_{33}\right)-1\right) = && \frac{1}{2}\left(e^{2\alpha-2\beta} - e^{2\gamma-2\beta}-1\right) \\
\Gamma^3_{12} =& \frac{1}{2}\left(\gamma^{33}\left(\gamma_{22}-\gamma_{11}\right)+1\right) && \frac{1}{2}\left(e^{2\beta-2\gamma} - e^{2\alpha-2\gamma}+1\right) \\
\Gamma^3_{21} =& \frac{1}{2}\left(\gamma^{33}\left(\gamma_{22}-\gamma_{11}\right)-1\right) = && \frac{1}{2}\left(e^{2\beta-2\gamma} - e^{2\alpha-2\gamma}-1\right)
\end{aligned}
\tag{257}
$$

with all others zero; note that the form of the rotation coefficients guarantees that only anisotropic parts of the metric tensor will have an effect on k^i (and therefore δ-terms, whether removable or non-removable always vanish in the geodesic equations; recall SECTION 9.1). Using the same method of conformally-related objects as described in ([23, part IIe]), define the vector $\bar{k}^\mu$: $a_F^2\bar{k}^\mu = k^\mu$ and the tensor $\bar{\gamma}_{ab} : a_F^2\bar{\gamma}_{ab} = \gamma_{ab}$; recall that $k^0_R = -k^i_R k^R_i = 1$. This gives us geodesic equations:

$$\frac{d\bar{k}^0}{d\lambda} + \frac{1}{2}\bar{\gamma}_{ab,0}\bar{k}^a\bar{k}^b = 0 \tag{258}$$

$$\frac{d\bar{k}^1}{d\lambda} + \left(e^{2\gamma-2\alpha} - e^{2\beta-2\alpha}\right)\bar{k}^2\bar{k}^3 = 0 \tag{259}$$

$$\frac{d\bar{k}^2}{d\lambda} + \left(e^{2\alpha-2\beta} - e^{2\gamma-2\beta}\right)\bar{k}^1\bar{k}^3 = 0 \tag{260}$$

$$\frac{d\bar{k}^3}{d\lambda} + \left(e^{2\beta-2\gamma} - e^{2\alpha-2\gamma}\right)\bar{k}^1\bar{k}^2 = 0 \tag{261}$$

. Despite the symmetry of these equations, their nonlinearity has inhibited the discovery of exact solutions and research into their properties is ongoing; see for example [24]. However, with solution up to quadratic order for the metric in hand (160,161,206,207), we can explicitly solve the equations in the case of weak waves. Let $\bar{k}^a = \bar{k}^a_R + \Delta\bar{k}^a(\lambda)$. Expanding out the geodesic equations to second order in the metric:

$$\frac{d\Delta\bar{k}^0_1}{d\lambda} + \frac{1}{2}\left[\alpha_1'\left(\bar{k}^1_R\right)^2 + \beta_1'\left(\bar{k}^2_R\right)^2 + \gamma_1'\left(\bar{k}^3_R\right)^2\right] = 0 \tag{262}$$

$$\frac{d\Delta\bar{k}^1_1}{d\lambda} + 2\left(\gamma_1 - \beta_1\right)\bar{k}^2_R\bar{k}^3_R = 0 \tag{263}$$

$$\frac{d\Delta\bar{k}^1_2}{d\lambda} + 2\left(\alpha_1 - \gamma_1\right)\bar{k}^1_R\bar{k}^3_R = 0 \tag{264}$$

$$\frac{d\Delta\bar{k}^1_3}{d\lambda} + 2\left(\beta_1 - \alpha_1\right)\bar{k}^1_R\bar{k}^2_R = 0 \tag{265}$$

$$\frac{d\Delta\bar{k}_2^0}{d\lambda}+\frac{1}{2}\left[\begin{array}{c}\left(\alpha_2'+2\alpha_1'\alpha_1\right)\left(\bar{k}_R^1\right)^2+2\bar{k}_R^1\alpha_1'\Delta\bar{k}_1^1+\\+\left(\beta_2'+2\beta_1'\beta_1\right)\left(\bar{k}_R^2\right)^2+2\bar{k}_R^2\beta_1'\Delta\bar{k}_1^2+\\+\left(\gamma_2'+2\gamma_1'\gamma_1\right)\left(\bar{k}_R^3\right)^2+2\bar{k}_R^3\gamma_1'\Delta\bar{k}_1^3\end{array}\right]=0 \tag{266}$$

$$\frac{d\Delta\bar{k}_2^1}{d\lambda}+2\left[\begin{array}{c}\left(\gamma_1-\beta_1\right)\left(\bar{k}_R^2\Delta\bar{k}_1^3+\bar{k}_R^3\Delta\bar{k}_1^2\right)+\\+\left(\gamma_2-\beta_2+3\gamma_1^2-3\beta_1^2\right)\bar{k}_R^2\bar{k}_R^3\end{array}\right]=0 \tag{267}$$

$$\frac{d\Delta\bar{k}_2^2}{d\lambda}+2\left[\begin{array}{c}\left(\alpha_1-\gamma_1\right)\left(\bar{k}_R^3\Delta\bar{k}_1^1+\bar{k}_R^1\Delta\bar{k}_1^3\right)+\\+\left(\alpha_2-\gamma_2+3\alpha_1^2-3\gamma_1^2\right)\bar{k}_R^1\bar{k}_R^3\end{array}\right]=0 \tag{268}$$

$$\frac{d\Delta\bar{k}_2^3}{d\lambda}+2\left[\begin{array}{c}\left(\beta_1-\alpha_1\right)\left(\bar{k}_R^1\Delta\bar{k}_1^2+\bar{k}_R^2\Delta\bar{k}_1^1\right)+\\+\left(\beta_2-\alpha_2+3\beta_1^2-3\alpha_1^2\right)\bar{k}_R^1\bar{k}_R^2\end{array}\right]=0 \tag{269}$$

. To first order, the equations are trivially solved by choosing $\lambda=\eta$ as the affine parameter; the problem of determining $d\lambda/d\eta$ is overcome by our choice of reference system, the lack of vector perturbations and the homogeneity of space:

$$\Delta\bar{k}_1^0=-\frac{1}{2}\left[\alpha_1\left(\bar{k}_R^1\right)^2+\beta_1\left(\bar{k}_R^2\right)^2+\gamma_1\left(\bar{k}_R^3\right)^2\right]_{\eta=\eta_E}^{\eta=\eta_R} \tag{270}$$

$$=-\frac{1}{2}\left[\tilde{\alpha}_1\left(\bar{k}_R^1\right)^2+\tilde{\beta}_1\left(\bar{k}_R^2\right)^2+\tilde{\gamma}_1\left(\bar{k}_R^3\right)^2+\frac{1}{3}\delta_1\right]_{\eta=\eta_E}^{\eta=\eta_R}$$

$$\Delta\bar{k}_1^1=2\bar{k}_R^2\bar{k}_R^3\int_{\eta_E}^{\eta_R}\left(\beta_1-\gamma_1\right)d\eta \tag{271}$$

$$\Delta\bar{k}_1^2=2\bar{k}_R^3\bar{k}_R^1\int_{\eta_E}^{\eta_R}\left(\gamma_1-\alpha_1\right)d\eta \tag{272}$$

$$\Delta\bar{k}_1^3=2\bar{k}_R^1\bar{k}_R^2\int_{\eta_E}^{\eta_R}\left(\alpha_1-\beta_1\right)d\eta \tag{273}$$

. The relationship (270) explicitly shows the quadrupolar nature of changes to the CMB alluded to in [19]. An unremoved removable perturbation changes the temperature of the whole sky isotropically; this confirms the effect noted by Hwang & Noh[42].

The equations for quadratic-order corrections read

$$\frac{d\Delta\bar{k}_2^0}{d\lambda}+\frac{1}{2}\left[\begin{array}{c}\left(\alpha_2'+2\alpha_1'\alpha_1\right)\left(\bar{k}_R^1\right)^2+2\bar{k}_R^1\alpha_1'\Delta\bar{k}_1^1+\\+\left(\beta_2'+2\beta_1'\beta_1\right)\left(\bar{k}_R^2\right)^2+2\bar{k}_R^2\beta_1'\Delta\bar{k}_1^2+\\+\left(\gamma_2'+2\gamma_1'\gamma_1\right)\left(\bar{k}_R^3\right)^2+2\bar{k}_R^3\gamma_1'\Delta\bar{k}_1^3\end{array}\right]=0 \tag{274}$$

which due to the cancellation of the terms in the right column integrates trivially to

$$\Delta\bar{k}_2^0=-\frac{1}{2}\left[\left(\alpha_2+\alpha_1^2\right)\left(\bar{k}_R^1\right)^2+\left(\beta_2+\beta_1^2\right)\left(\bar{k}_R^2\right)^2+\left(\gamma_2+\gamma_1^2\right)\left(\bar{k}_R^3\right)^2\right]_{\eta=\eta_E}^{\eta=\eta_R} \tag{275}$$

(reiterating the quadrupolar character of the change to the CMB, but generalizing it to anisotropic expansion); meanwhile for the space part of the vector

$$\frac{d\Delta\bar{k}_2^1}{d\lambda}+2\left[\begin{array}{c}(\gamma_1-\beta_1)\left(\bar{k}_R^2\Delta\bar{k}_1^3+\bar{k}_R^3\Delta\bar{k}_1^2\right)+\\+\left(\gamma_2-\beta_2+3\gamma_1^2-3\beta_1^2+2\delta_1\left(\beta_1-\gamma_1\right)\right)\bar{k}_R^2\bar{k}_R^3\end{array}\right]=0 \tag{276}$$

$$\frac{d\Delta\bar{k}_2^2}{d\lambda}+2\left[\begin{array}{c}(\alpha_1-\gamma_1)\left(\bar{k}_R^3\Delta\bar{k}_1^1+\bar{k}_R^1\Delta\bar{k}_1^3\right)+\\+\left(\alpha_2-\gamma_2+3\alpha_1^2-3\gamma_1^2+2\delta_1\left(\gamma_1-\alpha_1\right)\right)\bar{k}_E^3\bar{k}_E^1\end{array}\right]=0 \tag{277}$$

$$\frac{d\Delta\bar{k}_2^3}{d\lambda}+2\left[\begin{array}{c}(\beta_1-\alpha_1)\left(\bar{k}_R^1\Delta\bar{k}_1^2+\bar{k}_R^2\Delta\bar{k}_1^1\right)+\\+\left(\beta_2-\alpha_2+3\beta_1^2-3\alpha_1^2+2\delta_1\left(\alpha_1-\beta_1\right)\right)\bar{k}_R^1\bar{k}_R^2\end{array}\right]=0 \tag{278}$$

which has solutions

$$\Delta\bar{k}_2^1=-2\left\{\begin{array}{c}2\bar{k}_R^1\int_{\eta_E}^{\eta_R}\left[\left(\tilde{\gamma}_1-\tilde{\beta}_1\right)\left(\begin{array}{c}\left(\bar{k}_R^2\right)^2\int^{\eta}\left(\tilde{\alpha}_1-\tilde{\beta}_1\right)d\bar{\eta}+\\+\left(\bar{k}_R^3\right)^2\int^{\eta}\left(\tilde{\gamma}_1-\tilde{\alpha}_1\right)d\bar{\eta}\end{array}\right)d\eta\right]+\\+\bar{k}_R^2\bar{k}_R^3\int_{\eta_E}^{\eta_R}\left(\gamma_2-\beta_2+\tilde{\gamma}_1^2-\tilde{\beta}_1^2+2\tilde{\alpha}_1\left(\tilde{\beta}_1-\tilde{\gamma}_1\right)\right)d\eta\end{array}\right\} \tag{279}$$

$$\Delta\bar{k}_2^2=-2\left\{\begin{array}{c}2\bar{k}_R^2\int_{\eta_E}^{\eta_R}\left[\left(\tilde{\alpha}_1-\tilde{\gamma}_1\right)\left(\begin{array}{c}\left(\bar{k}_R^3\right)^2\int^{\eta}\left(\tilde{\beta}_1-\tilde{\gamma}_1\right)d\bar{\eta}+\\+\left(\bar{k}_R^1\right)^2\int^{\eta}\left(\tilde{\alpha}_1-\tilde{\beta}_1\right)d\bar{\eta}\end{array}\right)d\eta\right]+\\+\bar{k}_R^3\bar{k}_R^1\int_{\eta_E}^{\eta_R}\left(\alpha_2-\gamma_2+\tilde{\alpha}_1^2-\tilde{\gamma}_1^2+2\tilde{\beta}_1\left(\tilde{\gamma}_1-\tilde{\alpha}_1\right)\right)d\eta\end{array}\right\} \tag{280}$$

$$\Delta\bar{k}_2^3=-2\left\{\begin{array}{c}2\bar{k}_R^3\int_{\eta_E}^{\eta_R}\left[\left(\tilde{\beta}_1-\tilde{\alpha}_1\right)\left(\begin{array}{c}\left(\bar{k}_R^1\right)^2\int^{\eta}\left(\tilde{\gamma}_1-\tilde{\alpha}_1\right)d\bar{\eta}+\\+\left(\bar{k}_R^2\right)^2\int^{\eta}\left(\tilde{\beta}_1-\tilde{\gamma}_1\right)d\bar{\eta}\end{array}\right)d\eta\right]+\\+\bar{k}_R^1\bar{k}_R^2\int_{\eta_E}^{\eta_R}\left(\beta_2-\alpha_2+\tilde{\beta}_1^2-\tilde{\alpha}_1^2+2\tilde{\gamma}_1\left(\tilde{\alpha}_1-\tilde{\beta}_1\right)\right)d\eta\end{array}\right\} \tag{281}$$

.

14 Redshift and CMB variations

The geodesic of a light ray is related to its observed redshift by

$$z+1=\frac{\left(k^{\mu}u_{\mu}\right)_R}{\left(k^{\mu}u_{\mu}\right)_E} \tag{282}$$

[23]. Having determined $u_0=1$ and $u_i=0$ this simplifies to

$$z+1=\frac{a_F\left(\eta_R\right)}{a_F\left(\eta_E\right)}\bar{k}_R^0 \tag{283}$$

so, to quadratic order,

$$z+1\approx\frac{a_F\left(\eta_R\right)}{a_F\left(\eta_E\right)}\left\{1-\frac{1}{2}\left[\begin{array}{c}\left(\alpha_1+\alpha_2+\alpha_1^2\right)\left(\bar{k}_R^1\right)^2+\\+\left(\beta_1+\beta_2+\beta_1^2\right)\left(\bar{k}_R^2\right)^2+\\+\left(\gamma_1+\gamma_2+\gamma_1^2\right)\left(\bar{k}_R^3\right)^2\end{array}\right]_{\eta=\eta_E}^{\eta=\eta_R}\right\} \tag{284}$$

. Meanwhile, the temperature field

$$\frac{T_R}{T_E}=\frac{1}{z+1}\approx\frac{a_F\left(\eta_E\right)}{a_F\left(\eta_R\right)}\left\{\begin{array}{c} 1+\frac{1}{2}\left[\alpha_1\left(\bar{k}_R^1\right)^2+\beta_1\left(\bar{k}_R^2\right)^2+\gamma_1\left(\bar{k}_R^3\right)^2\right]+ \\ +\frac{1}{4}\left[\alpha_1\left(\bar{k}_R^1\right)^2+\beta_1\left(\bar{k}_R^2\right)^2+\gamma_1\left(\bar{k}_R^3\right)^2\right]^2+ \\ +\frac{1}{2}\left[\begin{array}{c}\left(\alpha_2+\alpha_1^2\right)\left(\bar{k}_R^1\right)^2+ \\ +\left(\beta_2+\beta_1^2\right)\left(\bar{k}_R^2\right)^2+ \\ +\left(\gamma_2+\gamma_1^2\right)\left(\bar{k}_R^3\right)^2\end{array}\right]\end{array}\right\}_{\eta=\eta_E}^{\eta=\eta_R} \tag{285}$$

so

$$\frac{\Delta T}{T_R}\approx\frac{a_F\left(\eta_E\right)}{a_F\left(\eta_R\right)}\left\{\begin{array}{c} \frac{1}{2}\left[\alpha_1\left(\bar{k}_R^1\right)^2+\beta_1\left(\bar{k}_R^2\right)^2+\gamma_1\left(\bar{k}_R^3\right)^2\right]+ \\ +\frac{1}{4}\left[\alpha_1\left(\bar{k}_R^1\right)^2+\beta_1\left(\bar{k}_R^2\right)^2+\gamma_1\left(\bar{k}_R^3\right)^2\right]^2+ \\ +\frac{1}{2}\left[\begin{array}{c}\left(\alpha_2+\alpha_1^2\right)\left(\bar{k}_R^1\right)^2+ \\ +\left(\beta_2+\beta_1^2\right)\left(\bar{k}_R^2\right)^2+ \\ +\left(\gamma_2+\gamma_1^2\right)\left(\bar{k}_R^3\right)^2\end{array}\right]\end{array}\right\}_{\eta=\eta_E}^{\eta=\eta_R} \tag{286}$$

.

15 Comparison with the observed CMB

Five-year and seven-year results[16, 18] from WMAP[30] give the best picture to date of the CMB. The WMAP observations reconfirm the constraint of the quantity $\Delta T/T<10^{-4}$[20]; any change to the CMB from acceleration must be equal to or smaller than this value in order to be compatible with observations, placing an additional constraint on cosmological models. This implies that in the current epoch, and in the absensce of further special alignment, $|\alpha|,|\beta|,|\gamma|\lesssim 10^{-5}$. In a matter dominated universe, under ordinary circumstances, this implies (since $\eta\lesssim 10^{-1}$; see SECTION 17)

$$\left|C_{\alpha 1,1}y_1^{\text{matter}}\right|\lesssim 10^{-5}\implies\left|C_{\alpha 1,1}\right|\lesssim 10^{-6} \tag{287}$$

$$\left|C_{\alpha 2,1}y_2^{\text{matter}}\right|\lesssim 10^{-5}\implies\left|C_{\alpha 2,1}\right|\lesssim 10^{-8} \tag{288}$$

; meanwhile in a radiation-dominated universe,

$$\left|C_{\alpha 1,1}y_1^{\text{radiation}}\right|\lesssim 10^{-5}\implies\left|C_{\alpha 1,1}\right|\lesssim 10^{-5} \tag{289}$$

$$\left|C_{\alpha 2,1}y_2^{\text{radiation}}\right|\lesssim 10^{-5}\implies\left|C_{\alpha 2,1}\right|\lesssim 10^{-6} \tag{290}$$

. The coefficients associated with the decaying mode are constrained to be smaller than those associated with the growing mode without further theoretical considerations.

15.1 CMB anomalies

Since the publication of the latest generation of CMB maps[28], numerous claims have been made (for example, [28, 32, 34, 39]) of anomalous structure in the CMB. While the WMAP team argue[17] that these phenomena are not of statistical significance, if a quasi-isotropic Bianchi IX universe could produce any of the perceived patterns it would point the way toward further observational studies of the CMB to determine cosmological parameters, and establish the quasi-isotropic Bianchi IX universe as a viable model for cosmology.

In all cases, we emphasize that the most likely explanation for any perceived pattern in the CMB which is not shown to be statistically significant is the null hypothesis: that is, the human perceptive phenomenon of pareidolia, the same phenomenon responsible for observing familiar shapes in clouds or the "Man in the Moon".

Cold spots, "fingers" and the "Axis of Evil"

Two compact, supposedly anomalous areas of low temperature have been noted in the CMB, the so called "cold spots".

The first of these (called Cold Spot I in [17]) is a region[31] covering approximately 15000 square degrees in the direction of the galactic center, much of which is 194 microkelvin[28] colder than the CMB mean temperature ($\Delta T/T_R = -7.12 \times 10^{-5}$).

Particularly noteworthy regarding Cold Spot I is its membership in one of four "fingers" spaced at roughly 90-degree angles around the galactic equator, intersticed by four areas of higher ($\Delta T/T_R = 7.12 \times 10^{-5}$) temperature[19]. Qualitatively, such a pattern is roughly consistent with the expected pattern if two of the functions $\alpha, \beta, \gamma > 0$ and if two of the the principle axes of the metric tensor lie on the axes of the cold and hot zones (implying the third axis points along the "Axis of Evil", see below). The so-called "Cold Spot II" reported by Vielva et al.[34, 37] also forms part of these "finger" structures[17].

Cold Spot I also has the angular size [19] predicts for the observed value of $\Omega_M \approx .3$.

Due to the coincidence of the cold spot with the direction of the galactic center, there are no optical observations in its direction (see FIGURE 5), and therefore there is no data on cosmic acceleration in the direction of Cold Spot I.

(EQUATION 286) implies that any cold spot resulting from anisotropy in the metric should be accompanied by an identical cold spot at a point antipodal to the original spot. Tegmark's examination[28] of the one-year WMAP data on the CMB low-order multipoles revealed an alignment between the CMB quadrupole and octupole in the direction of $(l, b) \approx (-110°, 60°)$ along which the quadrupole is nearly zero, an axis which Land & Maguiejo found[32] extended to the 16-pole and 32-pole as well; the alignment has been dubbed the "Axis of Evil". While examination of the three-year WMAP data[33] found the Axis of Evil to be of lower significance than initially thought (94%-98%), it still persists; the WMAP team's discussion of the alignment[17, pt. 7] admits the "remarkability" of this alignment and, while assigning its existence to chance, does not attempt to explain the "Axis of Evil" in full.

The Axis of Evil, which in equatorial coordinates[35, p. 43] lies close to RA 10:44 Dec +7.6°, falls within the zone in which redshift data has been collected for measurement of the cosmic

[19] The CMB dipole is defined as such a way as to be traceless, so $\int \Delta T_{\text{quadrupole}}/T_R dS = 0$.

deceleration parameter. To simplest linear approximation with a pure growing mode, (that is, that the functions α and α' are both small such that $\alpha^2 \approx 0$) this alignment rules out a CMB arising from cosmological gravitational waves as a source of cosmic acceleration. However, the fact of the alignment of the quadrupole, octopole, 16-pole and 32-pole indicates that non-linear contributions of gravitational waves to acceleration are not ruled out.

The question of the overall magnitude of the quadrupole, which is only 14% of the expected value[28, 38], has also been raised. The WMAP team[17, pt. 4] agree with Tegmark that the depressed quadrupole falls within the 95% confidence interval for simulations of the CMB, but do not attempt an explanation for the unusually strong octopole term. Long-wavelength gravitational waves can easily explain both through judicious choice the arbitrary constants $C_{\alpha 1,1}$ etc. in a manner compatible with the CMB. Efstathiou[29] supposes that the depressed quadrupole could be an indication of a closed universe; however the relationships he proposes generate zero contributions to the CMB power spectrum from the genuinely cosmological, intrinsic $n = 3$ waves found in Bianchi IX, and any observational test using his framework must rely on correct evaluation of gauge terms whose effective wavelengths must be far longer than the cosmic horizon. Furthermore, Efstathiou's conclusion that a closed universe would automatically require a scrapping of current inflationary models is contradicted by others; for example Guth argues that a universe that is closed but with a very large radius of curvature is not ruled out[40].

The quasi-isotropic Bianchi IX model cannot provide an explanation for hemispherical dipole asymmetry claimed by Ericksen *et. al.*[39].

16 Conclusions

The long-wavelength gravitational waves intrinsic to a quasi-isotropic Bianchi IX will cause a change in the cosmic microwave background with a distinctive quadrupolar signature. A radially-symmetric pattern of light deflections in the CMB resulting from shear may also be observed.

The almost-isotropic Bianchi IX model can be compatible with the CMB as observed, and can provide an explanation for perceived anomalies observed in the CMB by COBE and WMAP. However, the existence of these anomalies beyond the level of statistical noise is not certain; a possible route of cross-disciplinary research is open in the form of examination of the phenomenon of pareidolia as applied to the CMB.

Models of quasi-isotropic Bianchi IX relying on pure growing modes or pure decaying modes of the gravitational waves cannot simultaneously explain observed cosmic acceleration and the observed cosmic microwave background. Research into the non-linear regime of the Bianchi IX cosmology may elucidate the existence of a model of an accelerating universe in which isotropy is almost preserved.

Part V

An accelerating Bianchi IX universe preserving an almost-isotropic CMB

In order for a Bianchi IX universe to both appear nearly isotropic in the cosmic microwave background and to accelerate through the existence of long-wavelength gravitational waves, it must fulfill two conditions. The first is that the function $k^0(\eta_R)$ must have absolute value less than the limit imposed by observations of the cosmic microwave background, $\Delta T/T_R$. The second is that at least one of the functions $Q_a^b < 0$. It is possible for both these conditions to be simultaneously filled while remaining compatible with other observational constraints on cosmological parameters.

The idea of long-wavelength gravitational waves causing anisotropy in the CMB has been proposed, but not applied to the Bianchi IX universe. Grishchuk & Zel'dovich consider the possibility of long-wavelength gravitational waves existing in a Friedmann universe without violating the limits imposed by the CMB[41], but do not apply their work to the gravitational waves of cosmological character which appear in some homogeneous cosmologies. Campanelli *et. al.* suggest that such a universe could exist and propose a Taub-type Bianchi I universe which also includes anisotropic dark energy as an initial explanation for the observed CMB, complementing Rodrigues[113]. Critically, they do not consider gravitational waves as a generator of the anisotropy and treat the parameters of the Taub universe as if dark energy were simply established by fiat. Similarly, Kovisto and Mota[115] do not look beyond the Bianchi I model and instead fall back on exotic theories to explain dark energy.

17 Cosmological parameters

WMAP[18, 16] has produced an all-sky survey of the CMB which, if the universe is almost Friedmannian, can be used to constrain cosmological parameters.

Let the radius of curvature a_0 and conformal time η of the background Friedmann cosmology be treated as a free parameters; assume a closed universe. The WMAP seven-year data gives

$$H_0 = 70.4^{+1.3}_{-1.4}\text{km/s/Mpc} \tag{291}$$
$$\Omega_K = -.0025 \pm 0.0109 \tag{292}$$

(WMAP's analysis includes the value of Ω_K measured by baryon acoustic oscillations reported in [101]). The radius of curvature, Hubble parameter and curvature energy density are related by

$$a_0 = H_0^{-1}\sqrt{-\Omega_K^{-1}} \tag{293}$$

while the Hubble parameter, radius of curvature and η-time are related by

$$H_0 a_0 = \cot(\eta_0/2) \tag{294}$$

. Therefore we have limiting values (as defined by the 95% confidence boundary of the WMAP observations)

$$a_0 \geq 1.12 \times 10^{29}\text{cm} \tag{295}$$
$$\eta_0 \leq 0.0266 \tag{296}$$

and highest-confidence values

$$a_0 = 2.68 \times 10^{29}\text{cm} \tag{297}$$
$$\eta_0 = 0.00499 \tag{298}$$

. Meanwhile, the ratio of Hubble radius to radius of curvature is at least

$$H_0 a_0 \geq 8.67 \tag{299}$$

with a best-fit value of

$$H_0 a_0 = 20.0 \tag{300}$$

. In other words, if the universe is closed, then the cosmological gravitational waves of the Bianchi IX cosmology are of much, much longer wavelength than the observable universe.

Finally, from the value of the redshift of decoupling, $z_{\text{last scattering}} = 1090$, we can say by (19) that

$$\eta_R/\eta_E \approx 33.0 \tag{301}$$

. As the available data, including that from supernovae (see TABLE 1), does not exclude a flat universe, we are always free, in developing the theory of Bianchi IX and acceleration, to set the parameter η as close to zero as necessary; doing so will not, in and of itself, violate observations, but will instead be constrained by the impact of the decaying mode of the gravitational waves on the CMB.

18 Compatibility with the redshift

Of all the observed cosmological parameters observed by WMAP and other probes of the CMB, the ones that are directly observed are $\Delta T/T_R$ and $z_{\text{last scattering}}$. From these we can say that

in the current epoch the universe appears isotropic and that its expansion since last scattering has, on average to the present time, been isotropic; neither of these facts necessarily imply that the overall expansion was isotropic at any time before the present. Instead, the condition of quasi-isotropy simply implies that

$$\frac{dk^0}{d\eta} + \frac{1}{2}\gamma_{ab,0}k^a k^b \approx 0 \tag{302}$$

. This implies that shear is small, so

$$k^a \approx k_0^a \tag{303}$$
$$z + 1 \approx a_F(\eta_R)/a_F(\eta_E) \tag{304}$$

as in the background Friedmann case.

We can obtain a near-zero value to the wave functions in the present epoch by admitting the presence of both growing and decaying modes in the gravitational waves. We want the condition (assuming $\Delta T/T_R$ is positive; in the case that it is negative the inequalities must be reversed)

$$0 \leq \frac{a_F(\eta_E)}{a_F(\eta_R)} e^{\alpha(\eta_E)-\alpha(\eta_R)} \leq |\Delta T/T_R| \tag{305}$$

& similarly for β, γ. In its full form this equation is transcendental even when discussing weak waves, but expanding (155) to lowest surviving order in η, we obtain

$$\left|37C_{\alpha 1,1}\left(\eta_R^2 - \eta_E^2\right) + 4C_{\alpha 2,1}\left(\eta_R^{-3} - \eta_E^{-3}\right)\right| \leq |\Delta T/T_R| \tag{306}$$

. In a young universe, the times of emission and reception of a light ray are related by $\eta_E \approx \eta_R (z+1)^{-1/2}$ so

$$\left|37C_{\alpha 1,1}\left(1 - (z+1)^{-1}\right)\eta_R^2 + 4C_{\alpha 2,1}\left(1 - (z+1)^{3/2}\right)\eta_R^{-3}\right| \leq |\Delta T/T_R| \tag{307}$$

. Let:

- 10^{-g} be the amplitude of the growing mode $C_{\alpha 1,1}$, so $C_{\alpha 1,1} = \mathrm{sgn}(C_{\alpha 1,1})\, 10^{-g}$;
- 10^{-d} be the amplitude of the decaying mode $C_{\alpha 2,1}$, so $C_{\alpha 2,1} = \mathrm{sgn}(C_{\alpha 2,1})\, 10^{-d}$;
- 10^{-b} be the value of η_R;
- 10^{-T} be the value of $|\Delta T/T_R|$

so noting that $z \sim 1000 = 10^3$ our condition becomes approximately

$$\left|\mathrm{sgn}(C_{\alpha 1,1})\, 10^{-2b-g+3/2} - \mathrm{sgn}(C_{\alpha 2,1})\, 10^{3b-d+5}\right| \lesssim \left|10^{-T}\right| \tag{308}$$

. When the amplitude of the growing mode term dominates, this approximate inequality is satisfied by

$$-2b - g + 3/2 \lesssim -T \tag{309}$$

; when the decaying mode dominates, the inequality is satisfied by

$$3b - d + 5 \lesssim -T \tag{310}$$

. WMAP constrains $T \approx 4$ (the difference between lowest and highest temperatures is $2\Delta T_R/T = 1.4 \times 10^{-4}$) and $b \gtrsim 1$. This constrains the growing and decaying modes, when they act on their own, to:

$$g \gtrsim 7/2 \tag{311}$$
$$d \gtrsim 12 \tag{312}$$

. There exists a third possibility, in which the growing and decaying contributions are, in the current epoch, of equal size and opposite sign. For this to be the case, we need

$$-2b - g + 3/2 \approx 3b - d + 5 \tag{313}$$

; this approach relies on the observation of amplification of weak gravitational waves in rapidly-changing backgrounds (see SECTION 11.8). Since b is a free parameter this approximate equation can always be satisfied, but we still need to satisfy the constraints of the CMB.

19 Acceleration in the Bianchi IX universe

19.1 Order of magnitude estimates for gravitational wave amplitudes

Meanwhile, consider the tensorial deceleration parameter:

$$Q_1^1 \equiv -\frac{\ddot{a}_{11} a_{11}}{(\dot{a}_{11})^2} = \left[Q_0 - 2\frac{a_F}{\dot{a}_F}\dot{\alpha} - \frac{a_F^2}{\dot{a}_F^2}\left(\ddot{\alpha} + \dot{\alpha}^2\right)\right]\left(1 + 2\frac{a_F}{\dot{a}_F}\dot{\alpha} + \frac{a_F^2}{\dot{a}_F^2}\dot{\alpha}^2\right)^{-1} \tag{314}$$

& similarly for Q_2^2, Q_3^3; this relationship is exact. Evaluating (225) gives to lowest surviving order in η

$$\Delta Q^{(1)}_{11,\mathrm{growing}} \approx -70 C_{\alpha 1,1} \eta^2 \tag{315}$$

$$\Delta Q^{(1)}_{11,\mathrm{decaying}} \approx \frac{19}{2} C_{\alpha 2,1} \eta^{-1} \tag{316}$$

. With an observed $\Delta Q_{11} \approx -1$ we can write:

$$10^0 \approx \operatorname{sgn}(C_{\alpha 1,1})\, 10^{9/5-2b-g} - \operatorname{sgn}(C_{\alpha 2,1})\, 10^{1-d+b} \tag{317}$$

. In the case of the growing mode dominating we need $\operatorname{sgn}(C_{\alpha 1,1}) = +1$ and $9/5+2b-g \approx 0$. This forms a system of equations with (309) so we have, at the limit of the allowed CMB perturbation,

$$\begin{cases} 9/5 - 2b - g \approx 0 \\ -2b - g + 11/2 \approx 0 \end{cases} \implies \text{no solution} \tag{318}$$

; the growing mode cannot, on its own, cause the observed acceleration and be compatible with the CMB. For the decaying mode, we need $\operatorname{sgn}(C_{\alpha 2,1}) = -1$ and have

$$\begin{cases} 1 - d + b \approx 0 \\ 3b - d + 5 \approx -4 \end{cases} \implies \begin{cases} b \approx & -5 \\ d \approx & -6 \end{cases} \implies \begin{cases} \eta \sim 1 \times 10^5 \\ C_{\alpha 2,1} \sim 1 \times 10^6 \end{cases} \tag{319}$$

which is a nonsense result. Therefore neither the growing or decaying modes, on their own, can both cause observed acceleration and preserve the CMB. In the cases of the two modes having comparable effect on the metric and opposite sign, though, we can solve (317) with $\operatorname{sgn}(C_{\alpha 1,1}) = +1$, $\operatorname{sgn}(C_{\alpha 2,1}) = -1$ and

$$10^0 \approx -(C_{\alpha 1,1})\, 10^{9/5-2b-g} + (C_{\alpha 2,1})\, 10^{1-d+b} \tag{320}$$

$$\begin{cases} g - d \approx -5b - 7/2 \\ 9/5 - 2b - g \approx 0 \end{cases} \implies d \approx \frac{17}{10} + 7b, g \approx 2b - \frac{52}{10} \tag{321}$$

when the growing mode dominates the change in acceleration; this sets estimated limits on the parameters (since $b \gtrsim 2$):

$$C_{\alpha 1,1} \gtrsim 2 \times 10^1 \tag{322}$$
$$C_{\alpha 2,1} \lesssim 2 \times 10^{-16} \tag{323}$$

. When the decaying mode dominates the change in acceleration,

$$\begin{cases} g - d \approx -5b - 7/2 \\ 1 - d + b \approx 0 \end{cases} \implies g \approx -4b - 5/2, d \approx b + 1 \tag{324}$$

which constrains the parameters

$$C_{\alpha 1,1} \gtrsim 3 \times 10^{10} \tag{325}$$
$$C_{\alpha 2,1} \lesssim 1 \times 10^{-3} \tag{326}$$

. While the values for the growing mode are far greater than those for what could be called "weak" waves (recalling the constraints of SECTION 11.4), our educated estimate for $C_{\alpha 1,1}$ in the growing-mode dominated regime aligns nicely with the necessary strong-wave growing-mode value for ΔQ_1^1 disregarding the CMB. Therefore we can turn to an analysis in the quasi-isotropic regime.

19.2 Quasi-isotropic, strong growing mode acceleration

We apply the same reasoning as in the previous section, but we are aware of constraints (from [28]) not just on the CMB in the direction of the observed acceleration (which we continue to assign as the "α" or e_i^1 direction) but on the CMB in the other two (the "beta" and "gamma" directions):

$$\Delta T_\alpha/T_R + \frac{1}{2}\left(\Delta T_\beta/T_R + \Delta T_\gamma/T_R\right) = 7.1\times 10^{-5} \tag{327}$$
$$2\Delta T_\beta/T_R = 1.4\times 10^{-4} \tag{328}$$
$$2\Delta T_\gamma/T_R = 1.4\times 10^{-4} \tag{329}$$
$$Q_1^1 = -0.6 \tag{330}$$
$$\eta_R \lesssim 3\times 10^{-2} \tag{331}$$

.

In this and all regimes to follow we can also approximate $Q_F \approx Q_F^{\text{flat}} = 1/2$ to the limit of precision given the constraints on η; Q_F will be 1% stronger than Q_F^{flat} only when $\eta \approx 0.51$. Between the constraints (327-330) and the average over the sky of $\Delta T/T_R = 0$, we have four equations with seven unknowns (η, $c_0^\alpha, c_0^\beta, c_0^\gamma$, $C_{\alpha 2}, C_{\beta 2}, C_{\gamma 2}$). These equations are, explicitly (see equations 161, 225, 286):

$$7.1\times 10^{-5} \gtrsim (\eta_E/\eta_R)^2\left(e^{\alpha(\eta_E)-\alpha(\eta_R)} - 1\right) \tag{332}$$
$$1.4\times 10^{-4} \gtrsim (\eta_E/\eta_R)^2\left(e^{\beta(\eta_E)-\beta(\eta_R)} - 1\right) \tag{333}$$
$$1.4\times 10^{-4} \gtrsim (\eta_E/\eta_R)^2\left(e^{\gamma(\eta_E)-\gamma(\eta_R)} - 1\right) \tag{334}$$
$$Q_1^1 = \frac{Q_F - \tan(\eta_R/2)\,\alpha_R' - \tan^2(\eta_R/2)\,\alpha_R'' - \tan^2(\eta_R/2)\,\alpha_R'^2}{1 + 2\tan(\eta_R/2)\,\alpha_R' + \tan^2(\eta_R/2)\,\alpha_R'^2} \tag{335}$$

.

Trivially, we can see that in the limit of $\alpha, \beta, \gamma \to \infty$, we must have $Q_1^1 \approx Q_2^2 \approx Q_3^3 \to -1$; if acceleration is driven by growing modes of long-wavelength gravitational waves then in the long run the universe asymptotically approaches de Sitter expansion as if driven by a cosmological constant, indicating a solution in the regime of quasi-isotropy.

Consider the quasi-isotropic solution to the growing mode of the Einstein equations, normalized as in EQUATIONS (215-220). In the regime where c_0^α is sufficiently large that $A \gg 1$, we can approximate

$$c_2^\alpha \approx -\frac{1}{4}A^2 \tag{336}$$
$$c_2^\beta \approx c_2^\gamma \approx \frac{3}{20}A^2 \tag{337}$$

(an identical argument, with the functions α and β transposing their roles, applies for the case where $c_0^\alpha < 0$). From these terms we can also approximate the next order terms in the series:

$$c_4^\alpha \approx \frac{521}{5600}A^4 \tag{338}$$
$$c_4^\beta \approx c_4^\gamma \approx -\frac{15}{224}A^4 \tag{339}$$

. Approximating equation (335) to order $A^4\eta^4$ we obtain the relationships

$$Q_1^1(\eta_R) = \frac{Q_F + \frac{3}{8}A^2\eta_R^2 - \left(\frac{521}{1120} + \frac{1}{16}\right)A^4\eta_R^4}{1 - \frac{1}{2}A^2\eta_R^2 + \left(\frac{1}{16} + \frac{521}{1400}\right)A^4\eta_R^4} + \mathcal{O}\left(\left(\frac{1}{2}A\eta_R\right)^6\right)$$
$$Q_2^2(\eta_R) \approx Q_3^3(\eta_R) = \frac{Q_F - \frac{9}{40}A^2\eta_R^2 + \left(\frac{75}{224} - \frac{9}{400}\right)A^4\eta_R^4}{1 + \frac{3}{10}A^2\eta_R^2 + \left(\frac{9}{400} - \frac{15}{56}\right)A^4\eta_R^4} + \mathcal{O}\left(\left(\frac{1}{2}A\eta_R\right)^6\right)$$

. When $Q_1^1(\eta_R) = -0.6$ then $A\eta_R \approx 1.5 \pm 0.2$ ($c_0^\alpha \gtrsim 1.9$), within the limit of applicability of the expansion and also in the regime where the infinite series (212) converge. Thus, we have shown analytically that long-wavelength gravitational waves can explain cosmic acceleration if that acceleration is anisotropic.

We can also make the following qualitative assessments about acceleration. Firstly, its time-evolution is non-monotonic. In the α direction, the universe will at first exhibit slightly increased deceleration, before starting to accelerate. In the β and γ directions, deceleration will asymptotically increase toward infinity but then acceleration will decrease from infinity, quickly converging on the strong-field value of $Q_2^2 = Q_3^3 = -1$. Acceleration in the α direction begins at $A\eta \approx 1.2$ and the universe accelerates in every direction after $A\eta \approx 1.6$; thus the supposition that acceleration is a recent phenomenon is supported.

A universe that is accelerating in every direction is within the region allowed by the model. FIGURE (2) illustrates the evolution of the deceleration parameters as a function of time. The constraints placed on the decaying mode in (SECTION 17) and the upper limit on η_R show that the decaying mode of long-wavelength gravitational waves has not played a significant role in cosmic acceleration; in the epoch of last scattering, the deceleration parameter was almost isotropic and had a close to Friedmannian value.

We now turn our attention to the preservation of the CMB. We have three equations in three unknowns, taking the lowest term in the decaying mode and the lowest two terms in the growing

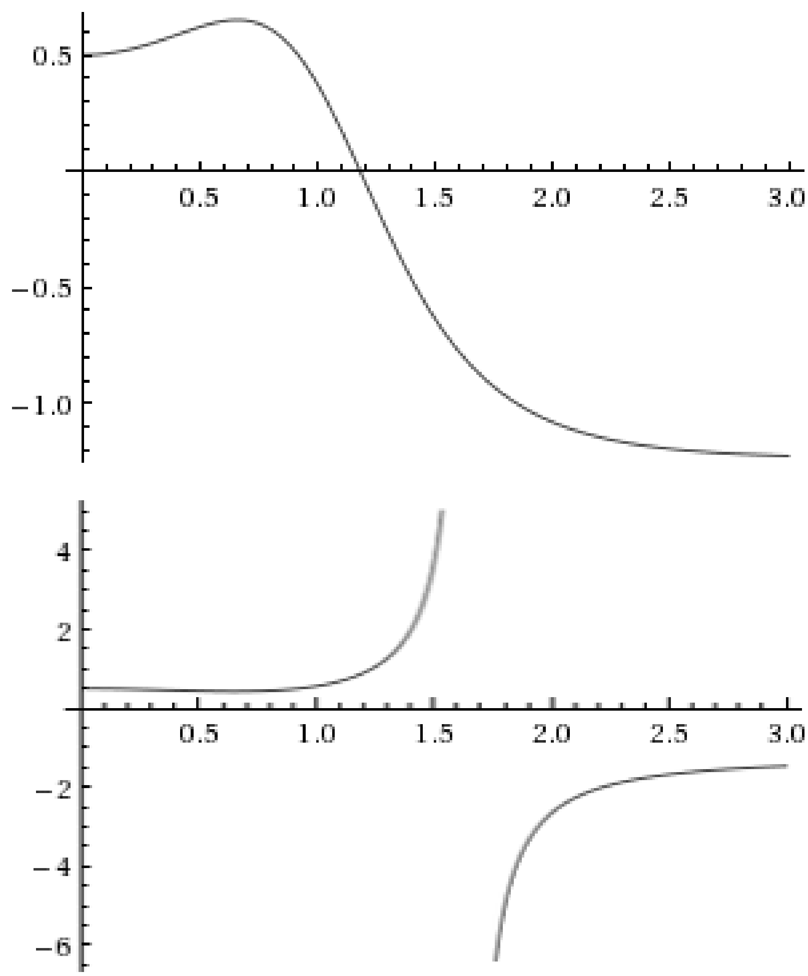

Figure 2: Deceleration parameter versus time

Along one direction, the universe at first decelerates, then quickly begins accelerating. Along the other two directions, the deceleration parameter goes to infinity before converging from negative infinity to the value -1. The vertical axis of each graph gives Q^b_a; the horizontal axis is in units of $A\eta$.

mode:

$$7.1\times10^{-5}\gtrsim 4\left(\eta_E/\eta_R\right)^2\left[\begin{array}{c} c_2^\alpha\left(\eta_E^2-\eta_R^2\right)+\frac{1}{2}\left(c_2^\alpha\right)^2\left(\eta_E^2-\eta_R^2\right)^2+\\ +c_4^\alpha\left(\eta_E^4-\eta_R^4\right)+C_{\alpha2,1}\left(\eta_E^{-3}-\eta_R^{-3}\right)\end{array}\right] \tag{340}$$

$$1.4\times10^{-4}\gtrsim 4\left(\eta_E/\eta_R\right)^2\left[\begin{array}{c} c_2^\beta\left(\eta_E^2-\eta_R^2\right)+\frac{1}{2}\left(c_2^\beta\right)^2\left(\eta_E^2-\eta_R^2\right)^2+\\ +c_4^\beta\left(\eta_E^4-\eta_R^4\right)+C_{\beta2,1}\left(\eta_E^{-3}-\eta_R^{-3}\right)\end{array}\right] \tag{341}$$

$$1.4\times10^{-4}\gtrsim 4\left(\eta_E/\eta_R\right)^2\left[\begin{array}{c} c_2^\gamma\left(\eta_E^2-\eta_R^2\right)+\frac{1}{2}\left(c_2^\gamma\right)^2\left(\eta_E^2-\eta_R^2\right)^2+\\ +c_4^\gamma\left(\eta_E^4-\eta_R^4\right)+C_{\gamma2,1}\left(\eta_E^{-3}-\eta_R^{-3}\right)\end{array}\right] \tag{342}$$

. As $30\eta_E\approx\eta_R$ we can further approximate

$$7.1\times10^{-5}\gtrsim\frac{1}{225}\left[\frac{1}{4}A^2\eta_R^2+\left(\frac{1}{32}-\frac{521}{5600}\right)A^4\eta_R^4+27000C_{\alpha2,1}\eta_R^{-3}\right] \tag{343}$$

$$1.4\times10^{-4}\gtrsim\frac{1}{225}\left[-\frac{3}{20}A^2\eta_R^2+\left(\frac{9}{800}+\frac{15}{224}\right)A^4\eta_R^4+27000C_{\beta2,1}\eta_R^{-3}\right] \tag{344}$$

$$1.4\times10^{-4}\gtrsim\frac{1}{225}\left[-\frac{3}{20}A^2\eta_R^2+\left(\frac{9}{800}+\frac{15}{224}\right)A^4\eta_R^4+27000C_{\gamma2,1}\eta_R^{-3}\right] \tag{345}$$

. If we take the inequalities as approximate equivalences and use $A\eta_R\approx1.5$ then this system has solutions

$$\begin{aligned} C_{\beta2,1}\approx C_{\gamma2,1}\approx &\quad -3\times10^{-7}\eta_R^3\\ C_{\alpha2,1}\approx &\quad -9\times10^{-6}\eta_R^3\end{aligned}$$

which is compatible with the estimates of (SECTION 19.1). That $C_{\alpha2,1}+C_{\beta2,1}+C_{\gamma2,1}\neq0$ is a consequence of the impossibility of *a priori* choosing an "unperturbed" temperature against which to compare anisotropic CMB fluctuations; the significance of non-linear terms means we cannot at the same time have the average over the whole sky of $\Delta T/T_R=0$ and have $\delta_1=0$, recalling (EQUATION 270).

We exhaust almost all the freedom in the system (332-335) in choosing to explain the "Axis of Evil" at the same time as acceleration; if this requirement is dropped and we treat CMB variations as insignificant then a broad range of solutions opens up. In particular, if the ratio of growing mode to decaying mode is approximately equal for all three of α,β,γ we always have sufficient freedom to choose a η that reduces CMB variation to below the level of detectability, at the expense of "tuning" the universe to place us as observers in the era when the CMB is nearly isotropic.

Compatibility with an almost-isotropic Hubble flow

The objection could be raised that the necessity of the universe contracting along two axes demands that a large region of the sky be blue-shifted, which would surely have been observed. This problem can be made to vanish into statistical noise by the choice of a sufficiently small η as (223) implies $a_F H_{11}=a_F'/a_F+\alpha'\approx2\left(\eta^{-1}+c_2^\alpha\eta\right)$ *etc.*

20 Conclusions

It is possible for a Bianchi IX universe with initial conditions $c_0^\alpha, c_0^\beta, c_0^\gamma \sim 1$ to display the acceleration observed in our universe while not only remaining compatible with the observed CMB but providing and explanation for potentially meaningful patterns in the CMB, specifically the so-called "Axis of Evil" and its associated phenomena such as cold spots. These conditions can be attained without additional constraints on the cosmological parameter of Ω_K, a parameter which is subject to further scrutiny and potentially tightening toward the flat universe case of $\Omega_K = 0$.

The method of combining strong growing modes with linear-order weak decaying modes of cosmological gravitational waves is borne out by observational data, which imply a difference of at least 17 orders of magnitude in amplitude between the growing and decaying modes. In the current epoch, decaying modes of cosmological gravitational waves can be neglected entirely. However, in the time close to last scattering, these modes may have participated at a strength comparable to the growing modes. Furthermore, the action of growing or decaying modes on their own is ruled out as an explanation for acceleration as neither on its own can preserve the CMB.

The question of how the ratio of growing mode to decaying mode is equal along all three principle axes of the metric tensor is answered easily if we postulate that cosmological gravitational waves present at the earliest moment in time were all in phase (the easiest way to do this is to postulate that they consisted of pure growing modes). As the functions α, β and γ would have all crossed the boundary from a $w = 1/3$ medium to a $w = 0$ medium at the same time, they would thus have remained in phase after last scattering, implying equal growing-to-decaying ratios for all three functions. As this transition happened in the very young universe ($\eta_E \lesssim 2 \times 10^{-3}$), the decaying mode that exists after last scattering would be very small.

The nonlinearity of Bianchi IX causes growing modes with initial values of order unity to develop exponentially and cause very powerful effects. The structure of the equations also indicates that multiple sets of initial conditions can lead to the same set of cosmological parameters. In light of the requirement of this model that both strong growing modes and weak but non-zero decaying modes of the gravitational waves exist, the possibility that these long-wavelength gravitational waves constituted the "pump field" of inflation[103] in the early universe should be explored.

The model proposed can be tested and falsified by observation of acceleration in areas of the sky 90° from the highly-observed field; in areas of the sky away from the currently-observed acceleration, we will see either a very large deceleration parameter or a negative one. From the analysis of acceleration data in (Part II) it is easy to see that, in the current state of observations, there are several possible areas of the sky where evidence of a gravitational-wave nature of cosmic acceleration could be sitting undetected.

Part VI

Conclusions

21 Directions for future research

The possibility of explaining cosmic acceleration through a Bianchi IX cosmological model opens up numerous possibilities for future research, both theoretical and observational.

While the difficulties with carrying out a full-sky optical survey of supernovae are understandable, experimental verification or falsification of a Bianchi IX model for acceleration requires nearly full sky coverage at high z to discover or rule out regions of anisotropy in the acceleration field. Infrared astronomy with wide sky coverage, for example WFIRST[65], presents the best possibility for these new observations through traditional astronomy. The Einstein telescope provides the tantalizing possibility of independent verification of the properties of dark energy through the examination of gravitational radiation.[111]

Meanwhile, the available supernova data can be re-examined for signs of acceleration, although given the comparatively small datasets in any particular area other than the highly-observed field and the equatorial bias in the distribution of the data this re-examination is less likely to produce definitive results. Célérier is justified in her criticisms[91] of the assumptions being made in proposed models of cosmological acceleration; it is curious that the authors if [89] reasoned, with 44 low-z sources, that "poor coverage at low and moderate Galactic latitudes [...] makes it practically impossible to distinguish between a peculiar monopole and a quadrupole" but that [1], which shares two authors with [89], does not even mention the possibility of cosmographic bias in its smaller sample of high-z sources.

Consideration should be given to the question of why cosmographic bias exists, and whether it points to an unexpected privileging of the observer: namely, the fact that modern observatories are hosted only in regions of the Earth that can afford to host them.

Perturbative methods for solving the Einstein equations for weak gravitational waves in Bianchi IX can be considered exhausted, having reached the limit of practical utility at quadratic order. Further analytic explorations should concentrate on the quasi-isotropic approach. The fact of Bianchi IX's easy reduction to a system of non-linear second-order ordinary differential equations combined with the divergence of Taylor series describing strong gravitational waves point toward either a Fourier-series approach or numerical methods for further analysis; the likelihood of chaotic behavior[73] in Bianchi IX, though, merits caution in the selection of initial conditions for any simulation.

Numerical examination of the quasi-isotropic regime should also be pursued for a fuller exploration of the space allowing for anisotropic acceleration while preserving an almost-isotropic cosmic microwave background. The next generation of microwave anisitropy probe should settle the question of whether the "Axis of Evil" and similar phenomena are genuine artifacts or statistical noise; in the meantime, the question of pareidolia in relation to the CMB has not been explored and deserves formal examination in order to raise awareness within the scientific community of the issue.

Overall, any theory is only as good as its ability to predict future results. Cosmic acceleration needs to be more closely examined, not only for time dependence, but for spatial dependence,

before any theory can emerge as preferred.

22 Implications of the Bianchi IX cosmological model

Since the discovery of cosmic acceleration, a wide range of scalar theories, ranging from the mundane to the exotic, have been put forward to explain the phenomenon. While the fact of acceleration, the discovery of which was the logical culmination of the hunt for the "missing mass" of the universe above and beyond that provided by dark matter, necessarily implies the slaughter of at least one sacred cow, the community of physicists has no consensus over which should be sacrificed the most readily.

Attempts to surrender homogeneity are physically the best-grounded but philosophically the most rash. Certainly the idea of a purely homogeneous cosmology is an approximation; but a universe which is not on average homogeneous, that is, where the homogeneous regions are rare exceptions, is one in which cosmology as a science ceases to be possible. The "Swiss cheese" universe has the advantage of making use of a known, exact solution to the Einstein equations and at least avoids the exceptionalism of the "Hubble bubble" proposal, but defeats itself on the grounds of testability.

Meanwhile, postulation of exotic states of matter has been done too enthusiastically for the evidence available. The simple fact of noting that the available data on acceleration was anisotropic exposes as irrational exuberance the rush to explain the phenomenon through the medium of a substance which has never been seen or even indicated in the laboratory, and whose theoretical justification is far beyond testability. The willingness of many to see acceleration as a falsification of the theory of general relativity looks all the more bizarre when counterposed with the unwillingness to explore gravitational-wave solutions to the problem.

The objection could be raised that asserting acceleration to potentially be anisotropic, in the weak sense of the word "isotropy", violates the cosmological principle by saying that our telescopes are privileged observers, in that our observational field happens to align with an axis of acceleration. This is no more so true than the "privilege" hypothesized by, for example, Riess *et. al.* when they assert, from a few dozen data points, that acceleration is a recent phenomenon, and that implicitly we are privileged observers in time for taking up cosmology just as the universe has begun to exhibit this behavior. While a cosmological constant is the simplest explanation for $w_X = -1$ on mathematical grounds, the lack of physical justification for a non-zero cosmological constant puts it in the same class as scalar-field theories. The simple fact is, $w_X = -1$ is, in the long run, the natural equation of state for any function which grows faster than the matter-driven terms in the background cosmology. The idea of the "Big Rip"[99], while intellectually (and emotionally) intriguing, makes the same mistake in the other direction, privileging observers to be alive just as the universe is beginning to tear itself apart. In this sense, a $w_X = -1$ field is the best preserver of the cosmological principle, and when the cosmological constant has been excluded the simplest explanation for acceleration comes from a tensorial field.

Similarly, when cosmic flatness is called into question – and it has never been and can never be definitively proven, it can only be disproven – the next-simplest model is the closed model. Recall that the Bianchi models are distinguished by their symmetries, and of all the Bianchi models with Friedmann universes as special cases, Bianchi IX has the most symmetric symmetries, obeying a "handedness" rule students learn before their first year of university. The fact of this "handedness" – parity – may even provide a neat explanation of the CP violation in particle physics[100], as Grishchuk alluded to[10].

The least speculative fact revealed by the assessment of available acceleration data is that more data is needed, from broader areas of the sky. The anticipated launch of WFIRST is likely to prove more momentous for cosmology than the flight of WMAP; WMAP largely reconfirmed what we already believed we knew, but WFIRST and SNAP will clearly illustrate how much we do not know. We also need techniques to see deeper into the sky and measure the distance-redshift relationship further into the past; the standard ladder of baryon acoustic oscillations[101] combined with better redshift data from WiggleZ may provide the necessary window.

That Bianchi IX could in principle contain accelerating regimes was never really in doubt. Numerical and qualitative analysis has indicated this ever since [10] noted that the vacuum equations contained a regular minimum, implying a positive first derivative for the Hubble parameter. The character of the acceleration has now been more properly investigated, bringing with it the possibility of a purely gravitational explanation for inflation, especially in light of the divergence of δ constructed only from growing modes in the radiation-dominated universe. An exploration of the differences between Bianchi I and Bianchi IX in a universe filled with ultra-relativistic matter could make Bianchi IX into a panacea for all the major problems of large-scale cosmology.

The unwillingness of the perturbed Bianchi IX cosmology to support decaying-mode gravitational waves stronger than linear order is puzzling, especially as the BKL universe always has a divergent term. The BKL universe, though, never reaches a singularity, and so the divergence of the a decaying mode never has time to take effect. Furthermore, the power law contraction along one axis could always be explained by a "growing" (non-diverging) function with negative coefficients, due to the exponential term in the metric.

The impact of strong waves on the CMB, meanwhile, also requires deeper explanation. Preservation of the CMB's apparent anisotropy at first glance appears to require some "tuning", a particular growing-decaying ratio which merits deeper questioning; there is also the outstanding matter of why we happen to live in one of the few periods of time when the CMB appears nearly isotropic. Clever examination of the symmetries of Bianchi IX may reveal a more satisfying answer, although the ability of Bianchi IX to explain CMB anomalies is one of its most satisfying features.

Most fundamentally, the biggest impact of the Bianchi IX theory of cosmic acceleration is the expansion of the cosmologist's parameter space. While in scalar models the only parameter truly open for discussion is the function describing the equation of state of dark energy, the gravitational waves of the Bianchi IX universe have four degrees of freedom; while the strength a non-zero cosmological constant has some theoretical justification in fundamental physics independent of large-scale cosmology, there is no immediately apparent reason why the gravitational waves in Bianchi IX should have any particular amplitude. As always in cosmology, we need more information than we have.

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Appendix

Table 3: Supernova observations used in analysis of acceleration

	Right ascension, J2000	Declination, J2000
Riess 1998 supernovae:[90]		
SN1994U	13:04:56	−6:3:39
SN1997bp	12:46:54	−10:21:27
SN1996V	11:21:31	2:48:40
SN1994C	07:56:40	44° 52’ 19”
SN1995M	09:38:42	−11:39:52
SN1995ae	23:16:56	−1:55:24
SN1994B	08:20:41	15:43:49
SN1995ao	02:57:31	−0:18:40
SN1995ap	03:12:28	0:41:43
SN1996R	11:16:10	0:11:39
SN1996T	10:05:28	−6:32:36
SN1997I	04:59:37	−2:50:58
SN1997ap	13:47:10	2:23:57
SDSS-II SNIa observations:[3]		
(Corner 1)	20:00:00	1:15:00
(Corner 2)	20:00:00	−1:15:00
(Corner 3)	04:00:00	1:15:00
(Corner 4)	04:00:00	−1:15:00
ESSENCE windows:[11]		
waa1	23:29:52.92	−08:38:59.7
waa2	23:27:27.02	−08:38:59.7
waa3	23:25:01.12	−08:38:59.7
waa5	23:27:27.02	−09:14:59.7
waa6	23:25:01.12	−09:14:59.7
waa7	23:30:01.20	−09:44:55.9
waa8	23:27:27.02	−09:50:59.7
waa9	23:25:01.12	−09:50:59.7
wbb1	01:14:24.46	00:51:42.9
wbb3	01:09:36.40	00:46:43.3
wbb4	01:14:24.46	00:15:42.9
wbb5	01:12:00.46	00:15:42.9
wbb6	01:09:00.16	00:10:43.3
wbb7	01:14:24.46	−00:20:17.1
wbb8	01:12:00.46	−00:20:17.1
wbb9	01:09:36.40	−00:25:16.7
wcc1	02:10:00.90	−03:45:00.0
wcc2	02:07:40.60	−03:45:00.0
wcc3	02:05:20.30	−03:45:00.0
wcc4	02:10:01.20	−04:20:00.0
wcc5	02:07:40.80	−04:20:00.0
wcc7	02:10:01.55	−04:55:00.0
wcc8	02:07:41.03	−04:55:00.0
wcc9	02:05:20.52	−04:55:00.0
wdd2	02:31:00.25	−07:48:17.3
wdd3	02:28:36.25	−07:48:17.3

wdd4	02:34:30.35	−08:19:18.2
wdd5	02:31:00.25	−08:24:17.3
wdd6	02:28:36.25	−08:24:17.3
wdd7	02:33:24.25	−08:55:18.2
wdd8	02:31:00.25	−09:00:17.3
wdd9	02:28:36.25	−09:00:17.3
HST supernovae:[14]		
SCP05D0	02:21:42.066	−03:21:53.12
SCP06H5	14:34:30.140	34:26:57.30
SCP06K0	14:38:08.366	34:14:18.08
SCP06K18	14:38:10.665	34:12:47.19
SCP06R12	02:23:00.083	−04:36:03.05
SCP06U4	23:45:29.430	−36:32:45.75
SCP06C1	12:29:33.013	01:51:36.67
SCP06F12	14:32:28.749	33:32:10.05
SCP05D6	02:21:46.484	−03:22:56.18
SCP06G4	14:29:18.744	34:38:37.39
SCP06A4	22:16:01.078	−17:37:22.10
SCP06C0	12:29:25.655	01:50:56.59
SCP06G3	14:29:28.430	34:37:23.15
SCP06H3	14:34:28.879	34:27:26.62
SCP06N33	02:20:57.699	−03:33:23.98
SCP05P1	03:37:50.352	−28:43:02.67
SCP05P9	03:37:44.513	−28:43:54.58
SCP06X26	09:10:37.888	54:22:29.06
SCP06Z5	22:35:24.967	−25:57:09.61
Riess "gold" dataset:[56, 54]		
Window 1	03:32:30	−27:46:50:00
Window 2	12:37:00	62:10:00

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